

Undermining the Cosmological Principle: Almost Isotropic Observations in Inhomogeneous Cosmologies

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We challenge the widely held belief that the cosmological principle is an obvious consequence of the observed isotropy of the cosmic microwave background radiation (CMB), combined with the Copernican principle. We perform a detailed analysis of a class of inhomogeneous perfect fluid cosmologies admitting an isotropic radiation field, with a view to assessing their viability as models of the real universe. These spacetimes are distinguished from FLRW universes by the presence of inhomogeneous pressure, which results in an acceleration of the fluid (fundamental observers). We examine their physical, geometrical and observational characteristics *for all observer positions* in the spacetimes. To this end, we derive *exact, analytic* expressions for the distance-redshift relations and anisotropies for any observer, and compare their predictions with available observational constraints. As far as the authors are aware, this work represents the first exact analysis of the observational properties of an inhomogeneous cosmological model for all observer positions. Considerable attention is devoted to the anisotropy in the CMB. The difficulty of defining the surface of last scattering in exact, inhomogeneous cosmological models is discussed; several alternative practical definitions are presented, and one of these is used to estimate the CMB anisotropy for any model. The isotropy constraints derived from ‘local’ observations (redshift $\lesssim 1$) are also considered, qualitatively. A crucial aspect of this work is the application of the Copernican principle: for a specific model to be acceptable we demand that it must be consistent with current observational constraints (especially anisotropy constraints) for all observer locations. The most important results of the paper are presented as exclusion plots in the 2-D parameter space of the models. We show that there is a region of parameter space not ruled out by the constraints we consider and containing models that are significantly inhomogeneous. It follows immediately from this that the cosmological principle cannot be assumed to hold on the basis of present observational constraints.

I. INTRODUCTION

In a previous paper [1] we found the complete class of irrotational perfect fluid cosmologies admitting an isotropic radiation field for every observer, and showed that they form a subclass of the Stephani family of spacetimes that includes the FLRW models, but, more importantly, also includes inhomogeneous cosmologies. Here we study the characteristics of these inhomogeneous spacetimes from all observer positions in order to assess their consistency with observations, especially with regard to isotropy constraints. In particular, we consider whether the FLRW models are the only viable candidates for a cosmological model. A pivotal part of this analysis is the cosmological principle (CP) and its relation to the Copernican principle and the observed isotropy of the universe.

Although it is an assumption based purely on philosophy, the Copernican principle is intuitively very appealing [2,3]: in order to reject an inhomogeneous cosmological model on the basis of its conflicting with the observed isotropy of the universe it is necessary to consider *all* observer positions and to show that *for most observers in that spacetime* the anisotropy observed is too large to be compatible with observations. Actually, in this paper we will, for simplicity, require consistency with observations for *all* observers – our results will thus be rather stronger than is strictly required by the Copernican Principle.

Having adopted the Copernican principle, the question then arises as to whether the observed isotropy of the universe, when required to hold at every point, forces homogeneity, thus validating the CP. Well, the nearby universe is distinctly lumpy, so it would be difficult to claim there is isotropy on that basis. However, the CMB is isotropic to

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one part in 10^3 . Together with the Ehlers, Geren and Sachs (EGS) theorem [4], or rather, the almost EGS theorem of Stoeger, Maartens, and Ellis [5] this allows us to say that within our past lightcone the universe is almost FLRW (i.e., almost homogeneous and isotropic), *provided* the fundamental observers in the universe follow geodesics (that is, as long as the fundamental fluid is dust). What the almost EGS theorem achieves is to provide support for the CP without the need to assume the (near) isotropy of every aspect of the spacetime: isotropy of the CMB alone is enough to ensure the validity of the CP (for geodesic observers). Since, however, the Stephani models of [1] do not satisfy the conditions of the almost EGS theorem (the acceleration is non-zero), its conclusions do not apply; indeed, we know from [1] that there are inhomogeneous spacetimes admitting an isotropic CMB (see also [6]).

A number of inhomogeneous or anisotropic cosmological models have been studied in relation to the CP. The homogeneous but anisotropic Bianchi and Kantowski-Sachs models [7] (see also [8] §6, and references therein) have been investigated with regard to the time evolution of the anisotropy. It can be shown, for example, that there exist Bianchi models that spend a significant phase of their evolution in a near-FLRW state, even though at early and late times they may be highly anisotropic (again, see §6 of [8] and references therein). Most interestingly for the work presented here, [9] show that there exist Bianchi models having an almost isotropic CMB but which are actually highly anisotropic (and have a large Weyl curvature, emphasising that they are certainly not FLRW), undermining the almost EGS theorem of [5].

Of the inhomogeneous models that arise in cosmological applications, by far the most common are the Lemaître-Tolman-Bondi dust spacetimes [10–12]. These are used both as global inhomogeneous cosmologies – probably the most important papers being [13,14], studying geometrical aspects, and [15–20] investigating observational aspects – and also as models of local, nonlinear perturbations (over- or under-densities) in an FLRW background [21–24]. See also [25] for a review. Non-central locations are rarely considered in the literature in the analysis of inhomogeneous cosmologies, owing to the mathematical difficulties that this usually entails, but [26] made a study of Tolman-Bondi spacetimes from a non-central location and applied their results to a ‘Great Attractor’ model. Most other non-central analyses, though, look only at perturbations of standard FLRW models.

However, there has been some consideration of Stephani solutions [27–31]. These are the most general conformally flat, perfect fluid solutions – and obviously therefore contain the FLRW models. They differ from FLRW models in general because they have inhomogeneous pressure, which leads to acceleration of the fundamental observers. [32] fitted a certain subclass of these models to the first type Ia supernova (SNIa) data of [33] using a low-order series expansion of the magnitude-redshift relation for central observers derived in [34], and found that they were significantly older than the FLRW models that fit that data. In [35] (see also [6]) we extend those results, using *exact* distance-redshift relations, and show that it would always be possible to find an acceptable fit to any data that could also be fit by an FLRW model (for plausible ranges of the FLRW parameters H_0 , Ω_0 and Ω_Λ). The best-fit models were again consistently at least 1 – 4 Gyr older than their FLRW counterparts, with models that provide the best fits to the newer SNIa data [36] giving an even greater age difference.

One feature of the Stephani models that has been the subject of much debate is their matter content. The usual perfect-fluid interpretation precludes the existence of a barotropic equation of state in general (because the density is homogeneous but the pressure is not), although a significant subclass – those with sufficient symmetry – can be provided with a strict thermodynamic scheme [1,37,38]. Individual fluid elements can behave in a rather exotic manner, having negative pressure, for example (cf. Sec. IID). For these reasons, amongst others, [39] has claimed that Stephani models are not a viable description of the universe, but [31, p.170], argues rather vigorously that this conclusion is incorrect, as do we. Cosmological models are often ruled out *a priori* because the matter fails to obey some or all of the energy conditions [40,41]. However, it has become increasingly difficult to avoid the conclusion that the expansion of the universe is accelerating, with the type Ia supernovae data of [36] being the latest and strongest evidence for this (see also [42,43]). This suggests that there is some kind of ‘negative pressure’ driving the expansion of the universe. For FLRW models, this must correspond to an inflationary scenario, with cosmological constant $\Lambda > 0$, or a matter content of the universe which is mostly scalar field (‘quintessence’ – see [44–47]; see also [48] for a discussion of the dynamical effects associated with Λ). Such matter inevitably fails to satisfy the strong energy condition. We show in Sec. IID that the subclass of Stephani models we consider contains some which satisfy all three energy conditions, and others which do not. In particular, we suggest in Sec. V that the models that would provide the best fit to the supernova data are models which also break the strong energy condition.

The main aim of this paper is to discuss the observational effects that arise in spacetimes admitting an isotropic radiation field when the observer is at a non-central location (the models we consider have spherical symmetry). Principally, we are interested in the anisotropy that this will introduce in observed quantities such as redshift, and we wish to compare these anisotropies with presently available observational constraints to see whether it is possible to rule out the inhomogeneous Stephani models on the basis of their anisotropy. We also examine the anisotropy of the CMB in these models: although they *admit* an isotropic radiation field, which is usually implicitly identified with the CMB, it cannot be assumed that decoupling will actually produce this radiation field in an inhomogeneous universe. We therefore suggest a number of ways to estimate the effect of inhomogeneity on decoupling, and determine

the resulting CMB anisotropy. Of course, we can always make any anisotropy as small as we like by assuming the observer is close to the centre of symmetry. But this is a very special position, so that such a resolution to the anisotropy problem would be in conflict with the Copernican principle (although it may be possible to circumvent this by invoking anthropic arguments [49]). In addition to the anisotropies, though, we also consider the constraints imposed by measurements of the value of the Hubble constant and the age of the universe, as well as constraints on the grosser features of the distance-redshift relations (see Sec. III C). While we do not perform fits to the available magnitude-redshift data here, we assume the results of [6,35], which show that it is possible to fit the observed SNIa data adequately with the models we consider.

In the following section we describe the spherically symmetric Stephani models and discuss their physical and geometrical properties, and we present in some detail the particular two-parameter subclass of the Stephani models that we will be studying. It will be shown that these models do not have particle horizons, in contrast to standard FLRW models. The energy conditions will be used to constrain the model parameters, leaving a manageable parameter set to be investigated further. Then, in Sec. II B, the transformation to coordinates centred on any observer will be derived. In Sec. III the various distance-redshift relations are presented for the models we consider, and observations are applied to constrain the values of the model parameters. The constraints we address are the value of the Hubble constant, age, the ‘size’ of the spatial sections (the meaning of which will be explained in Sec. III C) and, most importantly, the anisotropy of the CMB. We show that after applying these constraints there remains a region of parameter space containing models consistent with all of the constraints. Furthermore, we demonstrate in Sec. IV that many of the models not excluded by the constraints of Sec. III are distinctly inhomogeneous. In Sec. III E we examine the constraints on the acceleration and the inhomogeneity provided by ‘local’ (i.e., $z \lesssim 1$) observations. In two appendices we derive important results used in Sec. II.

Readers with families or short attention spans may wish to skip directly to Sec. VI, where the results are summarised and discussed.

II. THE SPHERICALLY SYMMETRIC STEPHANI MODELS

Stephani models are the most general conformally flat, expanding, perfect fluid spacetimes. They have vanishing shear and rotation, but non-zero acceleration and expansion. In [1] we showed that the irrotational perfect fluid spacetimes admitting an isotropic radiation field are a subclass of the Stephani models depending essentially on three free parameters and one free function of time. Although the general Stephani model has no symmetry at all, we only consider the class possessing spherical symmetry ($\mathbf{c} = \mathbf{0}$, or $\mathbf{x}_0 = \mathbf{0}$, in the notation of [1] – a full analysis of the models of [1] without spherical symmetry is given in [6]). The metric in comoving coordinates, from [1], is

$$ds^2 = \frac{(1 + \frac{1}{4}\Delta r^2)^2}{V(r, t)^2} \left\{ -c^2 dt^2 + \frac{R(t)^2}{(1 + \frac{1}{4}\Delta r^2)^2} (dr^2 + r^2 d\Omega^2) \right\}, \quad (1)$$

where c is the speed of light, $d\Omega^2 = d\theta^2 + \sin^2 \theta d\phi^2$ is the usual angular part of the metric and the function $V = V(r, t)$ is defined by

$$V(r, t) = 1 + \frac{1}{4}\kappa(t)r^2 : \quad (2)$$

$$\kappa(t) = 1 - R_{,t}(t)^2/c^2. \quad (3)$$

$R(t)$ is the scale factor, and $V(r, t)$ is a generalisation of the FLRW spatial curvature factor in isotropic coordinates (note that in [1] $R(t)$ is included in V). Since κ is a function of t the spatial curvature can vary from one spatial section to the next. In fact, it is possible for a closed universe to evolve into an open universe, or vice versa, in stark contrast to FLRW models [30].

We have already restricted ourselves to spherically Stephani models, but we need to reduce the parameter space further by introducing a form for $R(t)$ that depends on just a few parameters. To this end, we limit attention to the two-parameter family derived in Sec. IV of [51]. Thus we choose

$$R(t) = ct(at + b) \quad (4)$$

where a and b are the free parameters and

$$\Delta \equiv 1 - b^2, \quad (5)$$

so that (as can easily be seen by direct calculation)

$$\kappa(t) = \Delta - \frac{4a}{c}R(t). \quad (6)$$

We will henceforth refer to these models as Dąbrowski models. We require that after the big bang (and before any big crunch) $R > 0$, which forces

$$b \geq 0 \quad (7)$$

In (4)–(5) we retain factors of the speed of light, c , to facilitate comparison with the references given above and with observations. The units we will use are as follows: $[c] = \text{km s}^{-1}$, r is dimensionless, R is in Mpc and $[t] = \text{Mpc s km}^{-1} = [1/H_0]$, so that $[a] = \text{km s}^{-1} \text{Mpc}^{-1} = [H_0]$ and b is dimensionless. Note that these units are slightly different to those used in [34] because the parameters a and b in that paper contain a factor of c (so $[b] = [c]$, etc.). This explains the appearance of c in (4).

We will use T to denote the coordinate time of a specific epoch of observation along some observer's worldline (i.e., the coordinate age of the universe), again in Mpc s km^{-1} , and τ to denote *proper* time along a particular flow line. When we state ages they will generally be given in Gyr: $T[\text{Gyr}] \approx 978 T[\text{Mpc s km}^{-1}]$.

Formulae for the expansion and acceleration may be found in [1], and will be introduced as required, as will formulae for the energy density and pressure.

A. Geometry

The metric (1) is manifestly conformal to a Robertson-Walker metric with curvature Δ . If we multiply through by the conformal factor we see that the spatial sections really are homogeneous and isotropic, but the *actual* spatial curvature is time-dependent and is given by the curvature factor $\kappa(t)$ in $V(r, t)$. Thus, at any time t whether the spatial sections are closed, open or flat depends on the sign of $\kappa(t)$. If, at some point during the evolution of the universe, $t = -(b \pm 1)/2a$, then the curvature changes sign, as can easily be seen from (3) and (4). This does not happen in FLRW models, where the spatial curvature, k , is fixed. The distinction between the time-dependent true geometry (κ) and the fixed conformal geometry (Δ) should be borne in mind throughout what follows.

The metric (1) is not in its most advantageous form. The conformal geometry of the models is most easily studied by changing from the stereographic coordinate, r , to the ‘angle’ coordinate, χ (see equation (5.15) of [41]), appropriate to the value of Δ . Furthermore, for models with closed spatial sections (which will be our principal concern here) it is more convenient to choose a radial coordinate that is better able to reflect the fact that light rays can circle the universe many times. In such models the spatial surfaces have two centres of symmetry, $r = 0$ and $r = \infty$. Physically, there is nothing extraordinary about the point $r = \infty$: it is not infinitely far away from the centre, and it is quite possible for light rays to pass through it. This last point is particularly important for subsequent discussions, so we make the coordinate change (valid when $\Delta > 0$, although see below)

$$r = \frac{2}{\sqrt{\Delta}} \left| \tan \frac{\chi}{2} \right|. \quad (8)$$

Then $r \rightarrow \infty$ as $\chi \rightarrow \pi$. As a coordinate χ is restricted to the range $0 \leq \chi < \pi$. However, it will prove convenient to use χ not just as a coordinate but as a parameter along light rays. In the latter role its value can increase without bound as the rays circle the universe many times. Strictly speaking, we should distinguish these two uses, but it should not lead to confusion. The absolute value is taken in (8) so that, when χ increases beyond π , r remains positive. Using (8) in the metric (1) gives

$$ds^2 = \frac{1}{W(\chi, t)^2} \left\{ -c^2 dt^2 + \frac{R(t)^2}{\Delta} (d\chi^2 + \sin^2 \chi d\Omega^2) \right\}, \quad (9)$$

where (using (2 and 6)

$$W(\chi, t) = \cos^2 \frac{\chi}{2} V(r(\chi), t) = \cos^2 \frac{\chi}{2} + \frac{\kappa(t)}{\Delta} \sin^2 \frac{\chi}{2} = 1 - \frac{4aR}{c\Delta} \sin^2 \frac{\chi}{2}. \quad (10)$$

The conformal factor ($1/W$) is non-singular for all χ if $a \leq 0$. When $a > 0$ singularities $W = 0$ correspond to spatial and temporal infinity, and indicate that the universe has ‘opened up’. As the universe opens up and the sections become hyperbolic the coordinate χ represents a conformal mapping from a hyperbolic surface onto a sphere: spatial infinity will then correspond to some finite value of $\chi < \pi$. For a more detailed explanation of this see Theorems 4.1–4.4 of [30].

We can easily calculate the acceleration in these coordinates. It has only a radial (χ) component:

$$\dot{u}_\chi = -c^2 \frac{W_{,\chi}}{W} = \frac{2ac}{\Delta} \frac{R}{W} \sin \chi. \quad (11)$$

A simple calculation shows that the acceleration scalar, which we will need below, is just

$$\dot{u} \equiv (\dot{u}_a \dot{u}^a)^{1/2} = \frac{2|a|c}{\sqrt{\Delta}} \sin \chi \quad (12)$$

and is therefore time-independent (which, it turns out, is a necessary condition for a shear-free, irrotational perfect fluid to admit an isotropic radiation field – see [1]). Note that the units of \dot{u} are: $[\dot{u}] = [c/t] = [cH_0]$.

For completeness we mention that when the Dąbrowski model is conformal to an FLRW spacetime with hyperbolic geometry ($\Delta < 0$) the coordinate transformation is obtained from that just given by replacing trigonometric functions with their hyperbolic equivalent.

B. Non-Central Observers

Since the purpose of this paper is to provide insight into the observational characteristics of the Dąbrowski models for all observer positions, we must find expressions for the distance-redshift relations and other observable properties of the models from any point. For a general inhomogeneous metric this is far from trivial, but the Dąbrowski models have features that make this tractable (rather simple, actually). In particular, they are conformally flat. As has already been noted, in equation (9) the part of the metric in braces is exactly the form of the FLRW metric in ‘angle’ coordinates, so that the Dąbrowski models are manifestly conformal to the (homogeneous) FLRW spacetimes. This means that there is a group of transformations acting transitively on surfaces of constant time that *preserve the form of the FLRW part of the metric* (but not the conformal factor). If the spatial sections are closed, flat or open (in the conformal sense, i.e., according to the value of Δ), the transformations are rotations, translations or ‘Lorentz transformations’, respectively. After such a transformation the metric will have the form of an FLRW metric *centred on the new point*, multiplied by a modified conformal factor.

As will be shown in Sec. IID, we will be dealing exclusively with closed models ($\Delta > 0$), and so will concentrate on this case. To find the coordinate transformation to a non-central position, we perform a rotation of the spatial part of the metric, moving the origin ($\chi = 0$) to the point $\chi = \psi$ (ψ is the observer’s position in what follows). In appendix A we derive this transformation. The old χ is given in terms of the new (primed) coordinates by (A6). The conformal factor (10) then becomes (dropping the primes on χ and θ)

$$W \rightarrow W(\chi, \psi, \theta; t) = 1 - \frac{2aR(t)}{c\Delta} (1 - \cos \psi \cos \chi + \sin \psi \sin \chi \cos \theta), \quad (13)$$

while the rest of the metric retains its original form (but now in terms of the new coordinates). This transformation makes the study of our inhomogeneous models significantly easier, and allows us to find *exact* observational relations valid for any observer.

C. Lookback Time and the Horizon

If we calculate the lookback time in our models (i.e., the time, t , at which a galaxy at some position χ emits the light that the observer sees now at time T), which we can do directly from the metric (9) – see appendix B – we get, for any observer,

$$t(\chi) = \frac{bT}{(aT + b) \exp(b\chi/\sqrt{\Delta}) - aT}. \quad (14)$$

(Note that this function is continuous through $\chi = \pi$.) Now, $t \rightarrow 0$ if and only if $\chi \rightarrow \infty$. Thus, the whole of the big-bang surface is contained within the causal past of *every* observer in the spacetime and there is no horizon problem for the Dąbrowski models. This is in sharp contrast to FLRW universes: at early times the particle horizon is finite and contains only a small part of the big-bang surface, so that widely separated points can share no common influences. See Figs. 17 and 21 in [41].

D. Matter Content and Energy Conditions

The Stephani models do not have an equation of state in the strict sense, with the relationship between pressure and energy density being position dependent. Along each flow line, however, there is a relation of the form $p = p(\mu)$ – see [30]. The particular models we are using have an equation of state at the centre of symmetry (or everywhere in the homogeneous limit $a \rightarrow 0$) of the (exotic) form $p = -\frac{1}{3}\mu$. It was shown in [1], though, that these models do admit a (more general) thermodynamic interpretation (that is, temperature and entropy can be assigned in such a way that they are functions only of pressure and energy density). The matter content of these models with regard to the natural (comoving) velocity field is a perfect fluid with energy density

$$\frac{8\pi G}{c^2}\mu = \frac{3}{R(t)^2}, \quad (15)$$

and pressure given by

$$p = \mu c^2 \left(\frac{2}{3} \frac{V(r, t)}{1 + \frac{1}{4}\Delta r^2} - 1 \right) \quad (16)$$

$$= -\frac{1}{3}\mu c^2 \left(1 + \frac{8aR}{c\Delta} \sin^2 \frac{\chi}{2} \right) \quad (17)$$

$$= -\frac{1}{3}\mu c^2 \left(1 + \sqrt{\frac{24}{\pi G \mu}} \frac{a}{\Delta} \sin^2 \frac{\chi}{2} \right), \quad (18)$$

where (15) has been used to express the t -dependence of pressure in terms of the density. In other words, we have a position-dependent ‘equation of state’ of the form $p = -\frac{1}{3}\mu c^2 + \epsilon(\chi)\mu^{1/2}$. The appearance of $-\frac{1}{3}\mu$ as the dominant contribution (at early times, at least, when R is small so that, from (15), μ is large) immediately suggests a quintessential or scalar field interpretation of the matter [44–47], although it has been shown [52,53] that cosmic strings also give rise to this EOS. Although the energy-momentum tensor of the Stephani models has the perfect fluid *form*, the interpretation of the matter as an actual fluid is by no means required; other interpretations may also be valid, and the fact that there is no true EOS might even suggest a two-component interpretation. For the moment, though, this is as far as we will go to provide a physical motivation for the matter in the Dąbrowski models: in this paper we are only interested in the observational consequences of the geometry – although when we impose the energy conditions below we show that the matter is certainly not obviously unphysical.

We can see that there are singularities of density and pressure as $R(t) \rightarrow 0$ (i.e., at $t = 0, -b/a$), which correspond to the big bang and crunch for these models (the metric becomes singular at these points) – see [51]. We can also have a *finite-density* singularity, where only the pressure becomes singular. This happens when $r \rightarrow 2/\sqrt{-\Delta}$. Such infinite pressure is clearly not physical, so we can reject models with $\Delta < 0$. Models with $\Delta = 0$ are studied in [32], but it is difficult to compare them directly with $\Delta > 0$ models due to the different geometries of the spatial sections so we will not consider them in this paper, and we are left with $\Delta > 0$, i.e., $b < 1$ – see (5). As we explained in Sec. II the natural choice of positive $R(t)$ after the big bang ensures that $b \geq 0$.

Having calculated the pressure and density of the fluid we can now investigate its physical viability through the energy conditions. It is more convenient to use the original stereographic coordinates for this (i.e., the expression (16) for the pressure), since we wish to consider all values of Δ , until we find reasons to the contrary. The weak energy condition states that $\mu \geq 0$ and $\mu c^2 + p \geq 0$, whereas the strong energy condition requires in addition $\mu c^2 + 3p \geq 0$ (see, for example, [40] for a discussion). The weak energy condition rules out models with finite-density singularities (i.e., with $\Delta < 0$), because such models would contain regions (for $r > 2/\sqrt{-\Delta}$) where $\mu c^2 + p < 0$, as can easily be seen from (16). The weak energy condition also implies that $V \geq 0$, but this is always true since $V \rightarrow 0$ only at spatial (and temporal) infinity (even though the coordinates themselves may be finite). The strong energy condition, however, implies that $\kappa(t) \geq \Delta$ for all t . From (6) we can see that this is equivalent to $a \leq 0$ (since $R > 0$), so the models must have a big crunch ($R(t)$ is an ‘upside-down’ quadratic).

The dominant energy condition is more interesting. It states that $|p| \leq \mu c^2$, from which (15) and (18) immediately give

$$0 \leq \frac{1}{3} \frac{V(r, t)}{1 + \frac{1}{4}\Delta r^2} \leq 1.$$

The left inequality requires only that $\Delta \geq 0$, which rules out infinities in the pressure. The inequality on the right says that for all t and r

$$(\kappa(t) - 3\Delta)r^2 \leq 8 \quad (19)$$

must hold. This condition is always true for $a \geq 0$ (see (6)), as long as $\Delta \geq 0$. When $a < 0$, r is unbounded for $\Delta \geq 0$ (because κ is then positive – see (2)), so the left hand side of (19) must always be negative, $\kappa(t) \leq 3\Delta$. It is easy to see that $R(t)/c \leq -b^2/4a$, so, from (6), $\kappa(t) \leq 1$. Then $\kappa \leq 3\Delta = 3(1 - b^2)$ for all t provided

$$b \leq \sqrt{\frac{2}{3}} \approx 0.82. \quad (20)$$

For b larger than this the dominant energy condition will be broken at some time, in regions of the universe at large r . We will not consider further the intricacies of this. A glance at the exclusion diagrams, Figs. 9 and 10, shows that (20) does not eliminate a significant area of the allowed region. In light of this we will, for the moment, overlook (20) and investigate the properties of all models with $0 < b < 1$.

To summarise: we have used basic physical requirements, such as the occurrence of a big bang and the avoidance of pressure singularities, along with the energy conditions to restrict the ranges that the two parameters a and b (or Δ) can take. The results are:

$$a \leq 0, \quad 0 < b < 1 \quad (\text{i.e., } 0 < \Delta < 1) \quad (21)$$

(we reject $b = 1$ for simplicity, as explained above, and we refrain from invoking (20) until Sec. VI).

III. CONSTRAINING THE MODEL PARAMETERS USING OBSERVATIONS

So far we have considered only the ‘global’ physical properties of Dąbrowski models, but to really assess their potential viability as cosmological models it is necessary to confront them with observations. In this section we derive the distance-redshift relations that form the basis of the classical cosmological tests and compare them with available observational constraints to see whether any regions of parameter space are capable of providing a fit. We will impose constraints on the value of H_0 , age, size (the meaning of which will be explained below) and the anisotropy of the microwave background, leaving the wealth of data available from galaxy surveys and high-redshift supernovae for consideration in a future paper; the complexities involved in interpreting such data and applying it to idealised cosmological models require separate treatment.

Deriving the observational relations (redshift, angular size or area distance, luminosity distance and number counts) means relating the coordinates and metric functions to observable quantities. This requires knowledge of the observer’s motion (4-velocity), which can, strictly speaking, be specified independently of the background geometry. However, the Dąbrowski models contain perfect fluid, so we will identify the observer’s motion with the fluid velocity. We are not obliged to do this, and, given the strange form of the matter, it might be thought advantageous to instead assume that observers (i.e., galaxies) constitute a dust-like test fluid moving freely through the spacetime whose geometry is determined by the exotic matter. It will become clear in Sec. III D that if we were to make this assumption a large dipole anisotropy in the CMB would result (although the dipole in H_0 would be eliminated – see (30) with $\dot{u} = 0$) because such a flow will, in general, have a significant velocity relative to the Dąbrowski fluid flow, which, it will turn out, is very nearly in the rest frame of the CMB everywhere.

First we consider redshift. In general, it is no simple task to find analytic expressions for the redshift in any cosmological model; derivations usually rely on symmetries of the spacetime or other simplifying factors to solve the equations of null geodesics. Here we take advantage of the conformal flatness of Stephani models (or rather, of the fact that the Dąbrowski models are manifestly conformal to FLRW spacetimes), although we can also derive the redshift formula as a time-dilation effect; these procedures are outlined in appendix B. Using (B6) and (9) we find

$$1 + z(\psi, \chi, \theta) = \frac{R_0}{W_0} \frac{W(\psi, \chi, \theta; t)}{R(t)}, \quad (22)$$

where $R_0 = R(T)$, $W_0 = W(\psi, T)$ and t and χ are related by equation (14). Using (13) this is

$$1 + z = \frac{R_0}{W_0} \left\{ \frac{1}{R(t)} - \frac{2a}{c\Delta} (1 - \cos \psi \cos \chi) - \frac{2a}{c\Delta} \sin \psi \sin \chi \cos \theta \right\}; \quad (23)$$

showing that, for objects at any fixed χ , the inhomogeneity of universe manifests itself in the redshift as a pure dipole in angle around the sky ($\cos \theta$ term). This will be important in Sec. III D.

For metrics with spherical symmetry about the observer the angular size (and area) distance is given directly from the coefficient in front of the angular part of the metric, because symmetry ensures that for radial rays θ and ϕ are constant along the trajectory. For our models we do not have spherical symmetry about every observer, but the metric is everywhere conformal to a spherically symmetric metric, as can be seen from (9). Since null rays are not affected by the conformal factor they also remain at fixed θ and ϕ , so we can again obtain the angular size distance r_A from the coefficient of the angular part of the metric:

$$r_A(\psi, \chi, \theta) = \frac{R(t)}{\sqrt{\Delta W(\psi, \chi, \theta; t)}} |\sin \chi| \quad (24)$$

(again, χ and t are related by (14); the modulus signs around $\sin \chi$ ensure that (24) is valid even when χ is treated as a parameter along light rays and takes on values $> \pi$ – see Fig. 3). We can, for the first time, find the *exact* angular size-distance relation parametrically by combining equations (24) and (22), which bypasses the power-series method of [54]. This is valid for all sources seen from *any* observer position.

Luminosity distance r_L is related to r_A by the reciprocity theorem:

$$r_L = (1 + z)^2 r_A, \quad (25)$$

see [8,55]. This then allows the magnitude-redshift relation to be determined in the usual way: the apparent magnitude m of an object of absolute magnitude M is given in terms of the luminosity distance by

$$m - M - 25 = 5 \log_{10} r_L. \quad (26)$$

The task now is to limit a , b , and T using present observational constraints. A full discussion of each constraint is made in the following sections, and it is followed by exclusion diagrams showing the regions of parameter space for which a and b give a plausible cosmological model for *all* observer locations in these models.

A. The Hubble Constant

The expansion rate of the universe has been measured with reasonable accuracy. The Hubble constant is believed to lie in the range $50 \lesssim H_0 \lesssim 80 \text{ km s}^{-1} \text{ Mpc}^{-1}$, and we will use these limits to constrain the Dąbrowski models. The Hubble *parameter* for these models, which is related to the volume expansion Θ , is independent of position [1,31]:

$$H \equiv \frac{\Theta}{3} = \frac{R_{,t}(t)}{R(t)}, \quad H_0 = \frac{R_{,t}(T)}{R(T)}. \quad (27)$$

We can use this to place constraints on the time at which observations are made: for our models H decreases monotonically, so it will only lie in the observed range of H_0 for some range of T . For any observer with coordinate age T , we require

$$50 \lesssim \frac{2aT + b}{aT^2 + bT} \lesssim 80. \quad (28)$$

Usually, for simplicity, we will choose a specific value for H_0 (almost invariably that which produces the ‘worst case’). Then we can solve (27) for T .

However, when H_0 is actually *measured*, it is not necessarily equal to the expansion rate. What is measured in practice is the lowest order term in the magnitude-redshift relation, which gives the measured Hubble constant, H_0^m :

$$H_0^m = c^2 \frac{k^a k^b \nabla_a u_b}{(u_c k^c)^2} \Big|_0, \quad (29)$$

where k^a denotes the wave-vector of the incoming photons [55]. Equivalently, we can consider the gradient of the redshift-area distance curve at the observer (cf. [26]). If we measure the magnitude-redshift relation for objects in some direction, then, expanding $\nabla_a u_b$ in (29) in terms of the kinematical variables rotation ω_{ab} , shear σ_{ab} , acceleration \dot{u}_a and expansion Θ and using (27), we find (since $\sigma_{ab} = 0$ for Stephani models)

$$H_0^m(\theta) = H_0 - \frac{\dot{u}}{c} \cos \theta, \quad (30)$$

where \dot{u} is the acceleration scalar and θ is the angle between the acceleration vector and the direction of observation (which is opposite to the direction in which the photons are travelling). Since the acceleration is non-zero in the Stephani models there will be a dipole moment in H_0^m . The size of this in the Dąbrowski models is given directly from (12). If this is large in any model we can probably reject that model because a large dipole moment in H_0 is not observed. However, very nearby it is difficult to measure H_0 accurately due to peculiar motions and the discreteness of galaxies. There *is* a dipole moment in observations of somewhat more distant objects, which is assumed to be due to the fact that the Local Group is falling into the potential well produced by Virgo and the Great Attractor. The question is: what upper bound can be placed on the acceleration by observations? This issue was first raised in [1], and will be discussed in more detail in Sec. III E. For now we simply use (27) to constrain the epoch of observation, T .

B. The Age of the Universe

The original inspiration for [32] to study Stephani models was the potential resolution of the age problem that they provided, which at that time seemed to be virtually insurmountable within the framework of FLRW models (even when a non-zero cosmological constant was invoked): the high measured value of H_0 suggested an age, τ_0 , of at most about 11 Gyr for an FLRW cosmology, whereas globular cluster ages were thought to be up to 12-13 Gyr [59]. [32,35,6] showed that, for the particular Stephani models they considered, this apparent paradox disappears: the Dąbrowski models have ages that are consistently 1–4 Gyr older than their FLRW counterparts (for an observer at the centre of symmetry, at least). However, the age problem has recently been alleviated by a recalibration of the RR Lyrae distance scale and globular cluster ages in the light of Hipparcos [59], which has reduced the globular cluster ages considerably, to ~ 10 Gyr. The fit is still marginal, but the new ages are generally accepted as they allow a flat FLRW model to fit the observations provided that $H_0 \lesssim 67 \text{ km s}^{-1} \text{ Mpc}^{-1}$.

We certainly require, then, that at the epoch of observation our models are older than 10 Gyr. However, we will also consider the stronger constraint $\tau_0 > 12$ Gyr, partly to be conservative, but also because the diagrams for the 12 Gyr constraint are often clearer.

The age of the universe according to an observer at position ψ and at coordinate time T is simply the proper time elapsed from the big bang ($t = 0$):

$$\tau_0 = \int_0^T \frac{dt}{W(\psi, t)} = \frac{\Delta}{4|a|S\sqrt{1-b^2C^2}} \ln \left\{ \frac{\Delta - 2aTS(bS + \sqrt{1-b^2C^2})}{\Delta - 2aTS(bS - \sqrt{1-b^2C^2})} \right\} \quad (31)$$

(for $a \neq 0$ and $\psi \neq 0$, otherwise $W \equiv 1$ and $\tau_0 = T$), where $S = \sin(\psi/2)$, $C = \cos(\psi/2)$, and we take the value of T given by the solution of equation (27) as our constraint on the coordinate time for any specific H_0 .

In Fig. 1 the $\tau_0 = 12$ Gyr contours of the proper-age function are plotted for $H_0 = 60 \text{ km s}^{-1} \text{ Mpc}^{-1}$ and for several observer positions, ψ , showing how the age of an observer varies with ψ for different parameters a and b . The shaded regions contain models that are *less* than 12 Gyr old for at least one of the observer positions. This demonstrates that the proper-age of an observer is smallest at the antipodal centre of symmetry, $\psi = \pi$. Consequently, we will always use proper-age at $\psi = \pi$ to constrain the model parameters. We could weaken this constraint by requiring only that *most* observers are old enough, which would allow us to consider instead the age of observers at $\psi \leq \pi/2$ while still satisfying the Copernican principle (half of the observers would lie in this region). For simplicity, though, we will not do this here.

Finally, in Fig. 2 we show the age exclusion plot for the models (based on proper-age at $\psi = \pi$). We use three values of H_0 : 50, 60, and $70 \text{ km s}^{-1} \text{ Mpc}^{-1}$, and a proper-age of 10 Gyr (although the limits for 12 Gyr are also indicated). The shaded regions are excluded. It can be seen that unless we require the universe to be particularly old or the expansion rate high there is still a significant region of parameter space that cannot be excluded on the basis of age.

It should be noted here that these plots are meaningless for $b = 1$, because then $\Delta = 0$ and the model is conformal to an FLRW model with *flat* spatial sections, for which χ is not a good coordinate – as can easily be seen from (8).

C. Size and the Distance-Redshift Relation

When the spatial sections of a cosmological model are closed and there is no horizon problem light rays may circle the entire universe, perhaps many times. This is the case for the models we are considering here. What will the signature of this be in the various distance-redshift relations? The paths of light rays are determined by the conformal geometry of a spacetime, and it can be seen from (9) that our models are conformal to closed, static FLRW spacetimes.

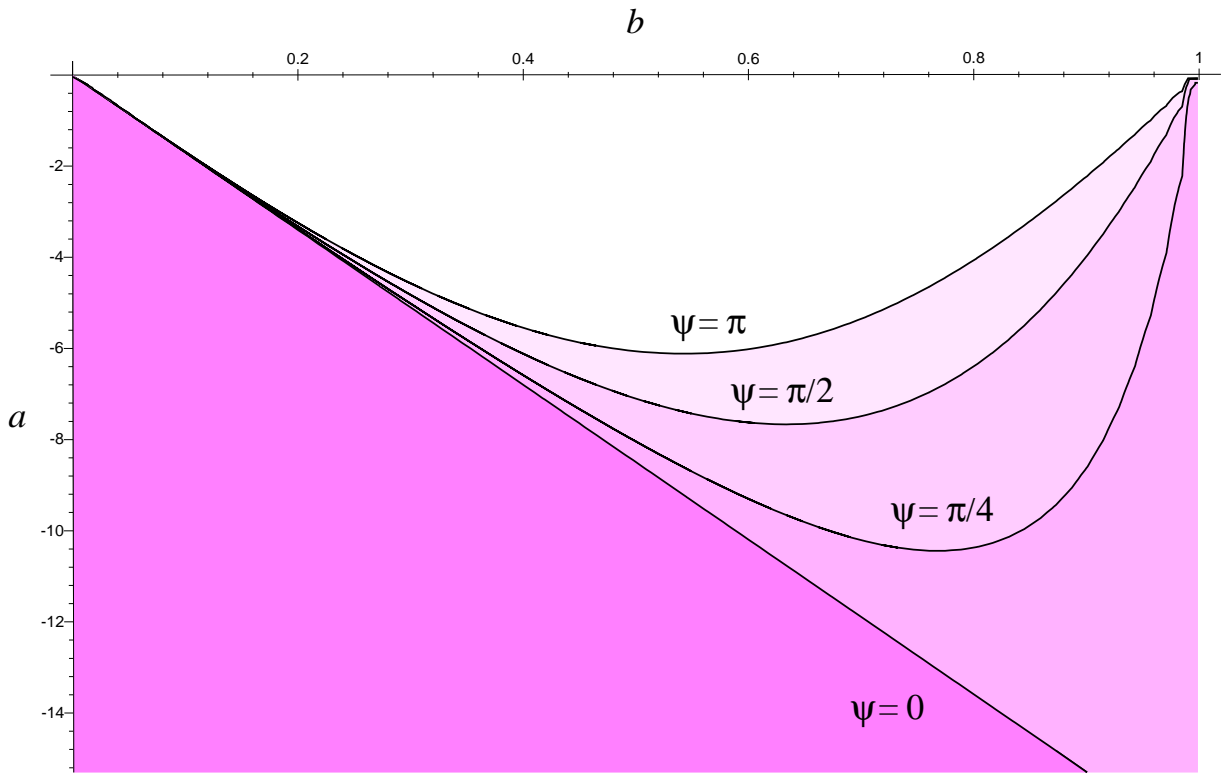


FIG. 1. $\tau_0 = 12$ Gyr exclusion plot for different observer positions, for $H_0 = 60 \text{ km s}^{-1} \text{ Mpc}^{-1}$. In the shaded areas the models are not old enough. $\psi = \pi$ is clearly the most restrictive case.

It follows that light rays from a point directly opposite the observer (i.e., from the antipode, $\chi = \pi$) will spread out around the universe isotropically from the antipode until they pass the ‘equator’ ($\chi = \pi/2$), where they will begin to converge and be focused onto the observer. As a result, a point source positioned exactly at the antipode will fill the entire sky when seen by the observer, so that its angular size distance, $r_A = (\text{physical length}/\text{apparent diameter})$, is zero. Similarly, the refocusing of light onto a point produces an infinite *flux* at the observer, and therefore the luminosity distance, r_L , is also zero ($m \sim \log_{10} r_L = -\infty$). It is obvious that whenever the light rays travel through a parameter distance χ that is an exact multiple of π , $r_A = r_L = 0$: this is reflected by the factor of $\sin \chi$ in (24).

This effect can be seen clearly in Figs. 3–5, where we show the two principal measures of distance as they vary with coordinate distance χ or redshift. Viewed as a function of χ , in Fig. 3, the zeros of the angular diameter distance occur at multiples of π for all model parameters. Looked at in terms of redshift, though (Fig. 4), it is clear that for small b the zeros are much closer together than for larger b , with the first zero occurring at $z \approx 1$ for $b = 0.25$. Figure 5 shows the luminosity distance-redshift relation, for comparison. These effects are not as unusual as they look, and can be found also in FLRW geometries for models with positive Λ – see §4.6.1 in [8] and references therein.

Can we rule out such apparently aberrant behaviour? Theories of structure formation are fairly well developed (see [60] for a thorough discussion and references), and the evolution of galaxies and the star formation rate (SFR), while not accurately known, are at least qualitatively understood. In particular, the SFR, which is very important for determining the luminosity of distant, young galaxies, is believed to fall off beyond $z \sim 2$ [61,62]. As a result, one could argue that there will be relatively few bright objects beyond some redshift z_{SF} that corresponds to the epoch at which galaxies ‘turned on’ and the SFR began to increase significantly. This would mean that the zeros in the distance-redshift relations would be essentially unobservable if they occurred at redshifts larger than z_{SF} because there would be no luminous objects to be seen magnified in the sky, whereas if the zeros occurred at lower redshifts than z_{SF} one could reasonably argue that there ought to be some signature of this in the observations. Since galaxies have only been observed (in the Hubble Deep Field, for example) with redshifts up to $z \sim 5$, and quasars have only been seen out to a similar redshift, we take $z_{SF} = 5$.

The constraints imposed by larger z_{SF} can be inferred from Figs. 6 and 7. (For example, if $z_{SF} \sim 30$, the redshift at which it is suggested that the very first luminous objects may appear [63], Fig. 7, gives $b \gtrsim 0.7$ for $a = -1$.) There are, of course, a number of factors that rather cloud these arguments, in particular the possible effects of extinction, which are not well known, as well as the fact that the number of objects ‘near’ to the antipode will be small, because

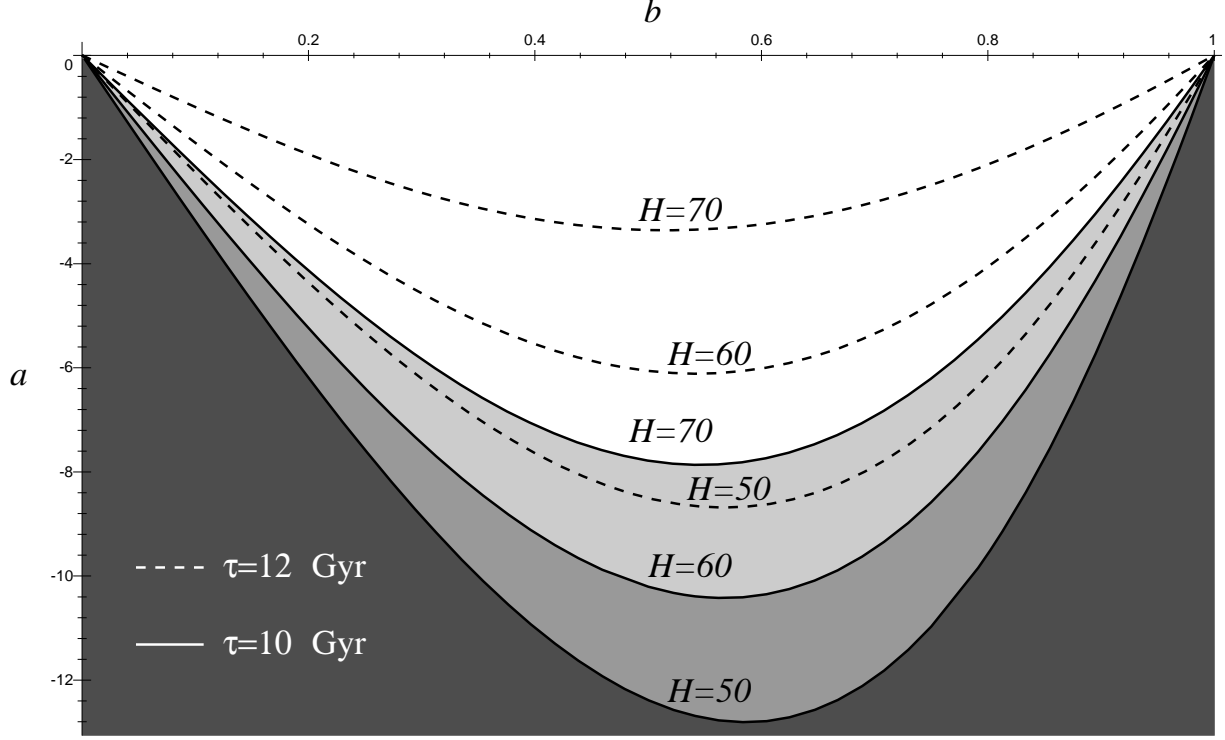


FIG. 2. The age exclusion diagram for various H_0 and proper age $\tau = 10$ Gyr. The shaded region represents the prohibited area. Also shown as dashed lines are the age limits for $\tau = 12$ Gyr. (Note that the region excluded for $H_0 = 70 \text{ km s}^{-1} \text{ Mpc}^{-1}$ contains the excluded regions for lower H_0 – the progressively darker shading indicates this.)

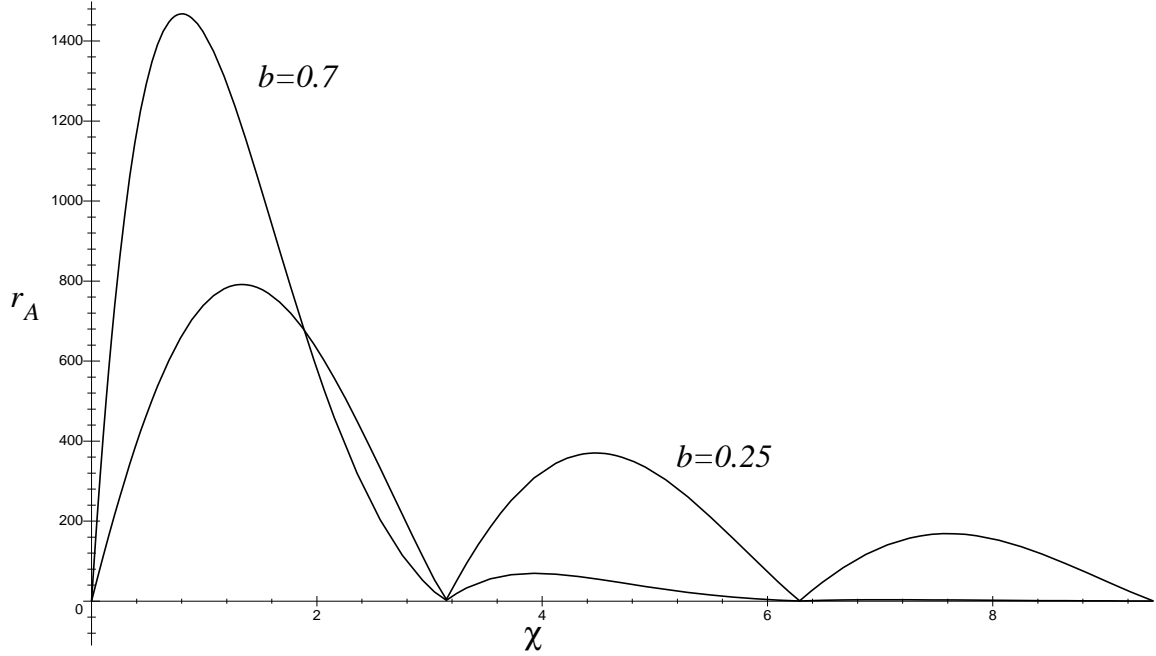


FIG. 3. Area distance from the centre as a function of χ for two values of b . $T = 15$ Gyr, $a = -1$.

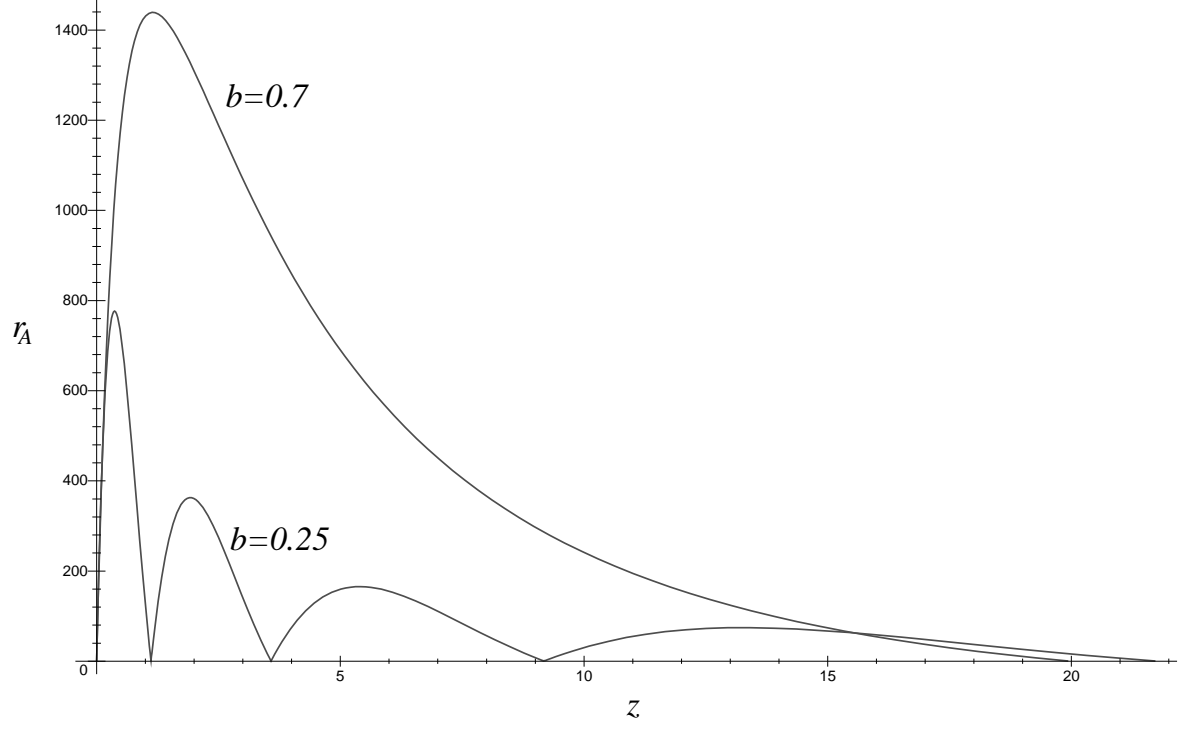


FIG. 4. Area distance from the centre as a function of redshift for the same parameters as Fig. 3. For small b the angular size distance oscillates far too rapidly.

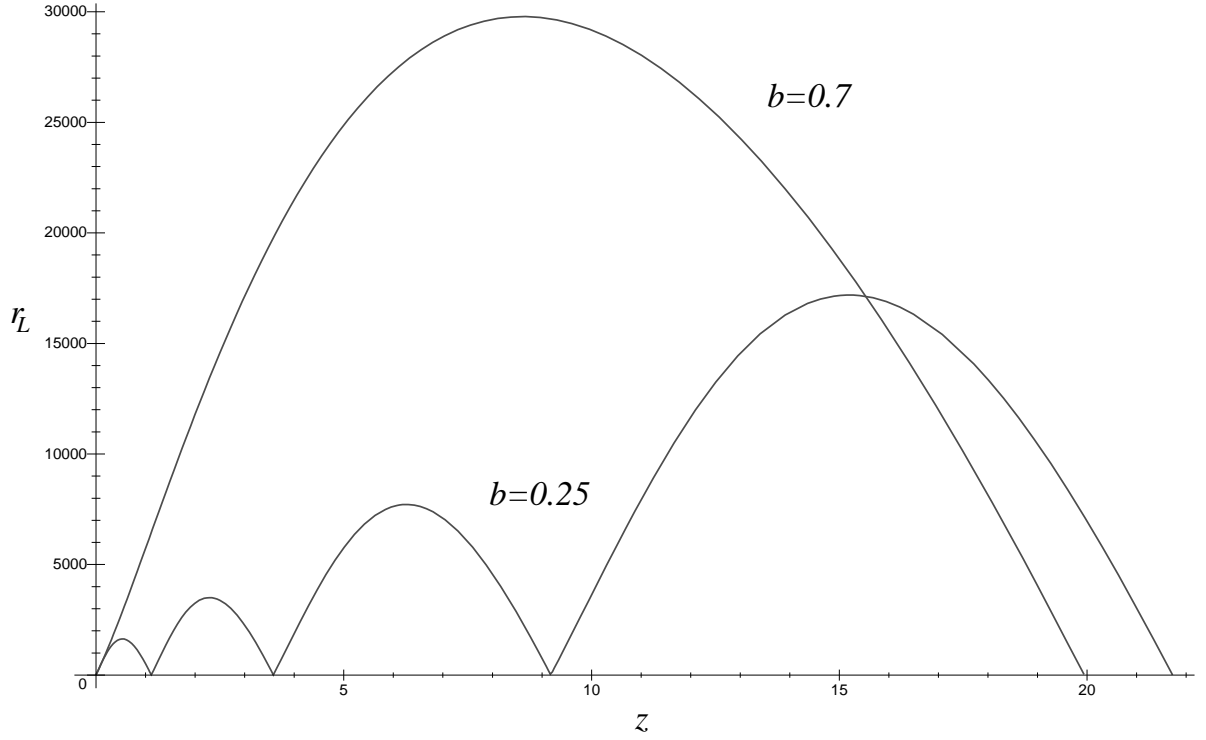


FIG. 5. Luminosity distance from the centre as a function of redshift for the same parameters as Figs. 3 and 4.

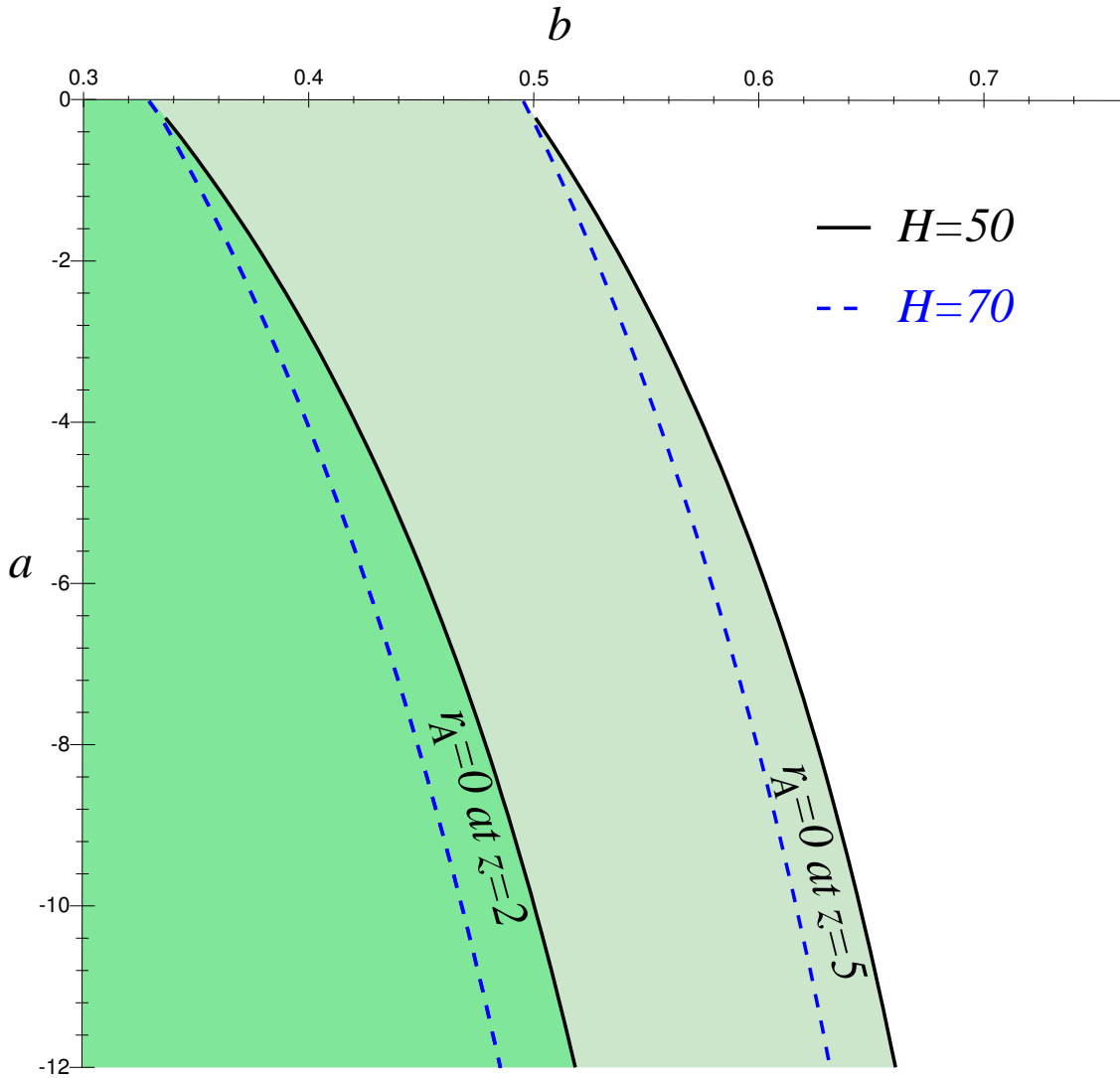


FIG. 6. Exclusion plot obtained by requiring that the first zeros of $r_A(z)$ occur at $z > z_\pi$, for $z_\pi = 2$ and $z_\pi = 5$. The shaded regions are excluded. Curves are given for $H_0 = 50$ and $H_0 = 70 \text{ km s}^{-1} \text{ Mpc}^{-1}$.

the spatial volume is diminished as a result of the geometry. Interestingly, number counts in the HDF show a dip at $z \sim 2$ [64,65] which would occur in any model for which the antipode is at this redshift. However, this dip is generally believed to be spurious [66].

If these arguments are not completely convincing, then at a simpler level the fact that the *observed* magnitude-redshift relation is known accurately out to $z \sim 1$ from type Ia supernovae [36], and is certainly not dipping down, allows us to say that there is no zero of luminosity distance below $z \sim 2$, say. We therefore also consider the constraint that results from requiring that there are no zeros below $z = 2$.

We wish, then, to constrain the parameters of our models by rejecting any models for which the first zero in the distance-redshift relations occurs at $z \leq z_\pi$, where $z_\pi = 2$ or $z_\pi = z_{SF} = 5$. Using (22), this means

$$1 + z(\chi = \pi) = \frac{R_0}{W_0} \frac{W(\psi, \pi, t_\pi)}{R(t_\pi)} > 1 + z_\pi, \quad (32)$$

where $R_0 = R(T)$, $W_0 = W(\psi, T)$ (the conformal factor at the observer) and t_π denotes the lookback time (14) at $\chi = \pi$. Again we determine the epoch of observation (i.e., the observer's coordinate time T) using (27). The solution of (32) for a and b is shown in Fig. 6 as an exclusion diagram. The effect of this constraint is to rule out small values of b , for any a . This is a reflection of the fact that, loosely speaking, b measures the ‘size’ of the universe: at early times the scale factor goes as bt , so that when b is small the spatial sections are small, light rays don’t take long to travel from antipode to observer, the scale factor changes relatively little during this time and the redshift of

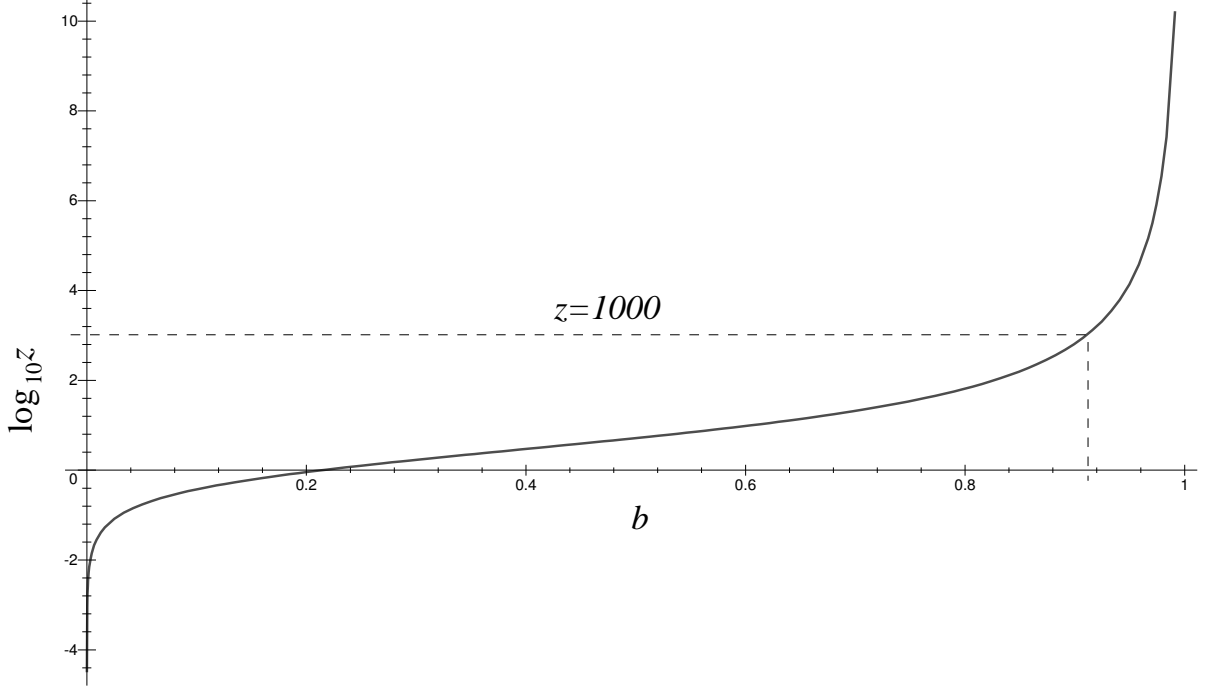


FIG. 7. Logarithmic plot of the redshift at which the first zero of r_A occurs as a function of b for $H_0 = 50 \text{ km s}^{-1} \text{ Mpc}^{-1}$ and $a = -1$. For the first zero to occur at $z > 1000$, so that recombination occurs ‘nearer to us’ than the antipode – i.e., at $\chi < \pi$ – requires $b \gtrsim 0.9$.

the antipode (which is dominated by $R_0/R(t_\pi)$ as in FLRW models) is small.

As a coda to this section we consider the effect of demanding that the first zero of r_A is effectively unobservable as a result of being ‘hidden’ behind the CMB. Figure 7 shows how the redshift of the first zero of $r_A(z)$ varies with b . If, instead of choosing $z_\pi = z_{SF}$ as our primary constraint, we want the first zero of $r_A(z)$ to happen at a redshift large enough for the universe to be opaque (i.e., before decoupling), then Fig. 7 shows that b must be quite close to unity. This figure also allows the extent to which values of b are excluded for any z_π to be estimated.

D. The Microwave Background Anisotropy

The CMB is observed today to be a blackbody at a temperature of $T_0 = 2.734 \pm 0.01 \text{ K}$, with a dipole moment of $T_1 = 3.343 \pm 0.016 \times 10^{-3} \text{ K}$ and quadrupole moment as large as $T_2 = 2.8 \times 10^{-5} \text{ K}$ (see [67] for details and references). It was emitted at a time when the radiation was no longer hot enough to keep Hydrogen ionised, causing it to decouple from matter, which happens at $T_{\text{dec}} \sim 3000 \text{ K}$. Idealised cosmological models do not have realistic thermodynamics (that is, they do not, in general, accurately describe the thermodynamic evolution of the gas and radiation mixture that fills the real universe). In FLRW models the epoch at which decoupling occurs is simply defined to be that corresponding to the redshift necessary to shift the temperature at decoupling to the observed mean temperature of the CMB, T_0 . From the redshift relation applied to the temperature of a blackbody (T will be used to denote temperature in this section),

$$T_{\text{obs}} = \frac{T_{\text{dec}}}{1+z}, \quad (33)$$

we infer that the CMB is formed at a redshift $z \approx 1000$. This definition is fine for homogeneous models, leading to a consistent definition of the time of decoupling for every observer at the same cosmic time, but raises an interesting point for the inhomogeneous Dąbrowski models, because the redshift depends on both the observer’s position, ψ , and the angle around the sky, θ . If we simply define the redshift of the CMB at any point to satisfy (33) with $T_{\text{obs}} = T_0$ then, by definition, we obtain a perfectly isotropic CMB for that observer, but we must choose a different emitting surface for each different observer. Such an observer-based definition of the CMB surface is clearly unsatisfactory.

In fact, we know already from the results of [1] that the models we consider *admit* an isotropic radiation field for every observer, which is implicitly identified with the CMB. However, this identification overlooks any physics underlying the production of the CMB. When, as with the original EGS theorem, the models under consideration turn out to be FLRW this is acceptable, since it can be assumed that the homogeneity applies to the production of the CMB: at some moment of cosmic time decoupling occurs everywhere throughout the universe. Unfortunately, when the models admitting an isotropic radiation field are inhomogeneous this is no longer acceptable, and some consideration must be given to the production of the CMB and how this may be affected by the different conditions at different places in the universe. To give detailed consideration to the physics of decoupling in the non-standard Dąbrowski cosmologies would take us beyond the scope of this paper, so instead we consider several alternative, pragmatic definitions of what is meant by the CMB surface, which, while not fully capturing the physics of decoupling, at least allow the influence of inhomogeneity at the time of decoupling to be estimated (it is important in these definitions to distinguish between the dominant exotic matter that is responsible for the geometry of the Dąbrowski models – see Sec. IID – and the putative ‘real’ baryonic gas that decouples):

1. If we avoid all consideration of the physics of decoupling, we *could* simply assume that it happens at such an early time that we could define the CMB to be free-streaming radiation ‘emitted at the big bang’, as is effectively assumed in the EGS and almost EGS theorems [4,5]. This only really makes sense if there is some natural definition of the radiation field at early times. For example, if the model is homogeneous and isotropic at early times, we can define a homogeneous and isotropic radiation field. Our models have exactly this property of homogeneity at early times (as can be seen from (10), $W \rightarrow 1$ as $t \rightarrow 0$). For our models this definition will result in a perfectly isotropic CMB for every observer [1] (although its observed temperature will be position dependent);
2. Ideally we would like to define decoupling in terms of the thermodynamics of the baryonic gas (at a fixed temperature $T \sim p/\rho$, say). However, the Dąbrowski matter is not an ideal gas and its pressure and density are not that of the real baryonic gas, so the utility of this definition here is limited;
3. In general inhomogeneous models we could choose a fixed value of some physical quantity such as density which would allow us to estimate the degree of inhomogeneity – for the Dąbrowski models this is equivalent to 4 because the density is homogeneous on cosmic time surfaces;
4. We could choose a surface of constant cosmic time, $t = t_{CMB}$ – since the Dąbrowski models possess a cosmic time coordinate with respect to which only the pressure is inhomogeneous this is a natural extension of the FLRW definition;
5. We could take a surface of constant *proper* time (based on the assumption of some common evolution for the ideal gas component at different positions) – see equation (31).

We will not consider 1 here for the reasons outlined above. Also, the homogeneity of Dąbrowski models at early times means that for small t the proper-age is virtually identical to the coordinate time (equation (31) with $W \approx 1$). It turns out that for times of observation that reproduce the observed H_0 any reasonable definition of the CMB surface puts it at an early time, which means definition 5 is virtually identical to 4. We will therefore define the CMB according to 4 in this section. This amounts to assuming that the early homogeneity allows for homogeneous physics on the CMB surface just as in FLRW models, and that the small inhomogeneity that is present affects only the redshift of points on the CMB surface. Note that the anisotropy of the CMB that arises from this definition of the CMB surface does not conflict with the results of [1]: a radiation field that is isotropic for every observer may still be defined, but any realistic process that gives rise to the CMB must reflect the inhomogeneity of the universe at the time of decoupling; the CMB anisotropy we derive in this section results essentially from the (small) inhomogeneity on the CMB surface.

It still remains, though, to decide exactly *which* surface of constant cosmic time the CMB originates from. Consider, for an observer at position ψ , the temperature distribution on the sky that the CMB would have if it were emitted from the surface $t = t_{CMB}$ (related by the lookback time formula (14) to some distance χ_{CMB}):

$$T_{\text{obs}}(\psi, \chi_{CMB}, \theta) = \frac{T_{\text{dec}}}{1 + z(\psi, \chi_{CMB}, \theta)}. \quad (34)$$

Equation (23) shows that we can write

$$1 + z(\psi, \chi_{CMB}, \theta) = 1 + z_0(\psi, \chi_{CMB}) + z_1(\psi, \chi_{CMB}) \cos \theta$$

where

$$\begin{aligned}
1 + z_0(\psi, \chi_{CMB}) &= \frac{R_0}{W_0} \left[\frac{1}{R_{CMB}} - \frac{2a}{c\Delta} (1 - \cos \psi \cos \chi_{CMB}) \right], \\
z_1(\psi, \chi_{CMB}) &= -\frac{2a}{c\Delta} \frac{R_0}{W_0} \sin \psi \sin \chi_{CMB} = \frac{\dot{u}(\psi)}{c^2} \frac{R_0}{\sqrt{\Delta} W_0} \sin \chi_{CMB}
\end{aligned} \tag{35}$$

(using (12) in the last equality and assuming $a \leq 0$). The mean redshift of the CMB surface is z_0 ; z_1 gives rise to an anisotropy in the CMB. We can therefore define the location of the CMB surface to be the t_{CMB} (or χ_{CMB}) that gives a mean redshift of 1000. That is, χ_{CMB} is the solution of

$$z_0(\psi, \chi_{CMB}) = 1000 \tag{36}$$

for any observer position ψ .

Having found χ_{CMB} we can evaluate the anisotropy in the temperature of the CMB. Since T_{obs} depends on the reciprocal of $1 + z$ the dipole moment in z will give rise to higher multipoles when expanded as a binomial series:

$$T_{\text{obs}}(\theta) = \frac{T_{\text{dec}}}{1 + z_0} \left[1 - \frac{z_1}{1 + z_0} \cos \theta + \left(\frac{z_1}{1 + z_0} \right)^2 \cos^2 \theta + O(\cos^3 \theta) \right], \tag{37}$$

that is,

$$\frac{\delta T(\theta)}{T} = -\frac{z_1}{1 + z_0} \cos \theta + \left(\frac{z_1}{1 + z_0} \right)^2 \cos^2 \theta + O(\cos^3 \theta).$$

The dipole moment of the CMB temperature is then (using (36))

$$\delta_1 = \frac{z_1}{1 + z_0} \approx 10^{-3} z_1. \tag{38}$$

Measurements of the CMB can now be used to constrain the model parameters. We at least require that the dipole moment should be no larger than the observed dipole anisotropy, $|\delta_1| < T_1/T_0 \approx 10^{-3}$ (i.e., $|z_1| < 1$). If this is satisfied for any model then it is clear that the quadrupole and higher multipole moments will all be $\lesssim 10^{-6}$ – certainly no larger than their observed values. In fact, such a constraint on z_1 is very weak, leaving vast tracts of parameter space entirely untouched. Moreover, there are very good reasons for believing that there is a significant contribution to the observed dipole moment from the peculiar velocity of the Local Group as a result of infall towards the Great Attractor [68], which can be measured with moderate accuracy using galaxy surveys, and seems to be consistent with the motion of the Local Group with respect to the CMB [69,70], although see [71]. Actually, it is not beyond the bounds of possibility that genuinely inhomogeneous background models such as those we consider here could mimic the grosser features of the local anisotropies (in particular the GA induced dipole effect – see Secs. III A and III E) as well as the CMB dipole. The ‘real’ universe would then be a perturbation of this inhomogeneous background. Viewed in this way, local observations (galaxy surveys, etc.) would reveal the effects of the large-scale inhomogeneity and also contain information about smaller-scale perturbations (peculiar velocities and the density contrast). That is, the peculiar velocity field and the density contrast that are inferred assuming an FLRW background would actually be thought of as containing one part that reflects the difference between the inhomogeneous and FLRW background models (and therefore contains the dipole effect referred to above, for example) and another part that is the ‘true’ perturbation. (The distinction between these two components is a fine one: ultimately it reflects the difference between linear and fully nonlinear perturbations of FLRW models.) We will consider this in a future paper [72]. However, we choose here to reject iconoclasm in favour of the more conservative viewpoint that most of the observed dipole *is* due to the peculiar motions induced by local inhomogeneities, but that there remains some leeway – up to 10% of the observed dipole – due to observational uncertainties, for there to be a purely cosmological contribution to the CMB dipole. Then the largest dipole moment that we can accept from our models is $|\delta_1| < 10^{-4}$, or

$$|z_1| < 0.1. \tag{39}$$

Given any model parameters and some observer position we adopt the following procedure. First we use (27) to determine the epoch of observation for some H_0 , as usual, then we solve for χ_{CMB} using (36). Having found all the parameters we need to determine z_1 we simply check (39) to see whether the model, or at least that observer position, must be rejected. In practice we can simply solve (39) as an equality to obtain a as a function of b at the boundary of the allowed region, and this is what is shown in Fig. 8 (for $\psi = \pi/2$), where it can be seen that a low value of H_0 constrains our models most – in contrast to the age constraint. This is because H_0 decreases monotonically with time,

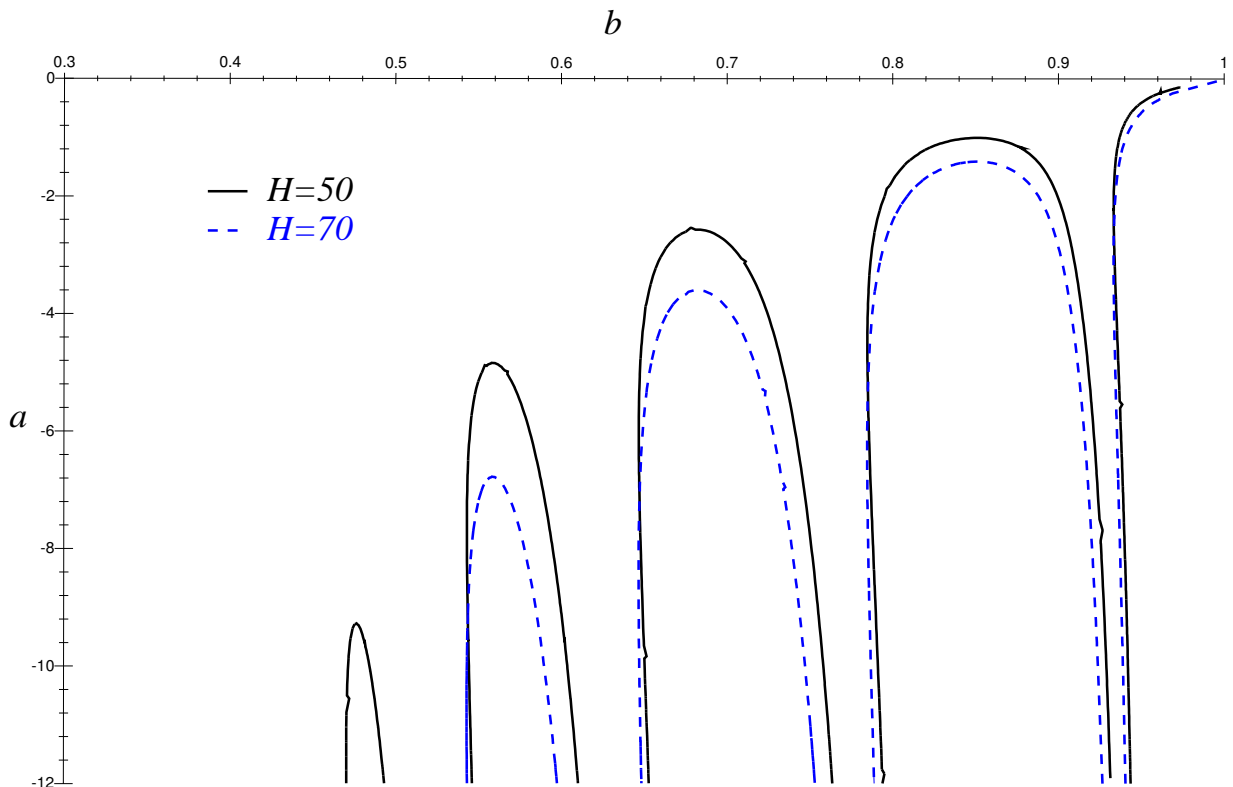


FIG. 8. The exclusion diagram from CMB anisotropies for an observer at $\psi = \pi/2$. The curves for $H_0 = 50$ and $70 \text{ km s}^{-1} \text{ Mpc}^{-1}$ are shown. As H_0 increases, the ‘fingers’ move down to more negative a . The allowed region lies between and above the fingers.

so small H_0 corresponds to a later time of observation and therefore a later time for the CMB surface, which means that W has evolved to become more inhomogeneous. We choose $\psi = \pi/2$ because, as is clear from (11) and (35), the anisotropy is generally worst there, so if a model is rejected at $\psi = \pi/2$ it will be unacceptable everywhere. Again, therefore, for models not excluded in Fig. 8 detection of a CMB anisotropy of the magnitude that we observe would be typical, and the Copernican principle need not be abandoned for these models to be viable.

The finger-like excluded regions in Fig. 8 appear because for different model parameters χ_{CMB} takes on different values, and for some parameters this value is very close to (a multiple of) π , so that the entire CMB seen by an observer in such a model is emitted from virtually a single point (the antipode). Since redshift depends only on the relative conformal factors at emitter and observer for our conformally flat models, the CMB must be almost exactly uniform however inhomogeneous the model (i.e., whatever the value of a).

E. The Local Dipole Anisotropy

Although we are not in a position to use real observations to constrain the dipoles that would be detected in observations of the ‘local’ universe (in galaxy surveys, for example, where $z \lesssim 0.01$, or with type Ia supernova data, for which $z \lesssim 1$), we can at least consider these effects qualitatively. It will turn out that for the models we consider observations of the dipole variation in the distance of objects at a given redshift (or the dipole in z at a given distance) directly constrain the acceleration of the fundamental observers. In fact, it is clear from (30) that this is the case for *any* model (at low redshift at least), so that measurements of these dipole moments at low z permit the acceleration \dot{u}^a to be measured. It is then only a question of the tightness of the constraints imposed by real observations, and the extent to which the acceleration signal can be separated out from other components in the peculiar velocity field. This we consider in [72]. Note that the acceleration dipole grows with distance at low z , as can be seen from (30), so that it is distinct from the dipole resulting from GA infall, which amounts to an overall Lorentz boost by a constant factor and is therefore independent of distance (see [1]). It should be noted that this is only true when objects more distant than the GA are observed; closer in the dipole structure due to GA infall is more complicated, and at very small distances there should be no dipole variation in H_0^m (to first order), because $\dot{u} = 0$ in the standard interpretation

(although there will be a quadrupole component due to shear).

In this section we adopt the null hypothesis that the universe *is* well described by a (perturbed) FLRW model (i.e., that observations are at least consistent with $\dot{u} = 0$), and determine whether, and under what circumstances, local observations may provide a tighter constraint on the Dąbrowski models than the CMB (Fig. 8).

From (23) and (12) the redshift dipole for objects at any coordinate radius χ from an observer at ψ (in (12) χ is the observer position relative to the centre of symmetry, which we now denote ψ) is seen to be

$$|z_1(\psi, \chi)| = \frac{\dot{u}(\psi)}{c^2} \frac{R_0}{\sqrt{\Delta} W_0} |\sin \chi| \quad (40)$$

(cf. equation 35). For small z this just corresponds to the H_0^m dipole in (30), since $z = (H_0^m/c)r_{\text{prop}}$ (at low z the distance measures r_A and r_L are the same as proper distance r_{prop}). It is clear that the redshift anisotropy for objects at some distance from the observer is closely related to the acceleration \dot{u} of the fundamental observers and the dipole moment in the measured H_0^m in (30). Defining $\delta H = \dot{u}/c$, the dipole in redshift for objects at a given χ becomes

$$|z_1(\psi, \chi)| = \frac{\delta H}{c} \frac{R_0}{\sqrt{\Delta} W_0} |\sin \chi|. \quad (41)$$

If we assume that the time of observation, T , is fairly close to $t = 0$ (as is generally the case for the values of H_0 we allow), then models with $a < 0$ have $T \ll -b/2a$ (the time at which the scale factor reaches its maximum value and $H_0 = 0$) and $W_0 \approx 1$, so that $R_0/c \approx bT$ and $R_{t_0}/c \approx b$. Equation (41) then becomes, with the help of (27),

$$|z_1(\psi, \chi)| \approx \frac{R_{t_0}}{c\sqrt{\Delta}} |\sin \chi| \frac{\delta H}{H_0} \approx \frac{b}{\sqrt{\Delta}} |\sin \chi| \frac{\delta H}{H_0}. \quad (42)$$

Note that dependence on distance from the observer only arises through the $\sin \chi$ factor, so for objects at any given distance from the observer

$$|z_1(\psi, \chi)| \lesssim \frac{b}{\sqrt{\Delta}} \frac{\delta H}{H_0}. \quad (43)$$

In principle, therefore, local observations of redshift anisotropies determine the H_0 -dipole, and therefore the acceleration \dot{u} , through (40) and (41). Conversely, any measurement of the acceleration or Hubble dipole can be used to constrain the dipole anisotropy z_1 at all redshifts through (42) or (43).

Before considering the constraints imposed by local observations, we compare (43) with the CMB constraint (39) imposed in Sec. III D. If we assume that the bound in (39) is reached, i.e., that the CMB dipole due to the inhomogeneity of the Dąbrowski model we are considering is $|z_1| = 0.1$, then it follows from (43) that

$$\frac{\delta H}{H_0} \geq 0.1 \frac{\sqrt{\Delta}}{b} \quad (44)$$

($\sqrt{\Delta}/b$ is a decreasing function of b , so that this constraint is stronger for smaller b). That is, if the universe was well described by a Dąbrowski model and the cosmological CMB dipole was measured to be as large as the bound specified in (39), then the local H_0 -dipole must be larger than $0.1\sqrt{\Delta}/b$. (For example, a variation in H_0 of less than 20% can only arise if $\sqrt{\Delta}/b \leq 2$, or $b \geq 1/\sqrt{5} \approx 0.45$; for a variation of less than 10% we must have $b \geq 1/\sqrt{2} \approx 0.71$.) Since the CMB anisotropy constraint (39) allows models with $b \gtrsim 0.5$ (see Fig. 9), corresponding to a variation in H_0 around the sky of at least 17% for models on the boundary of the allowed region in Fig. 9 where $|z_1| = 0.1$, if present observations show that the H_0 -dipole is less than 17% some of the models allowed by (39) will be excluded by these local observations. Note, though, that (44) is considerably stronger than is really required for most model parameters, owing to the fact that the $\sin \chi$ factor in (42) was neglected. This amounts to adopting as the CMB constraint the envelope of the fingers in Figs. 8, 9 and 10, which would obviously overlook large areas of parameter space that should really be allowed.

Although it would seem that the easiest way to constrain the acceleration in the Dąbrowski models would be to measure the local H_0 -dipole, it is not really possible to measure $\delta H/H_0$ accurately [72]. Moreover, there is known to be a dipole in observations of galaxies at somewhat larger redshifts ($z \sim 0.01$), which is usually interpreted as the effect of infall of the Local Group towards the Great Attractor [68,69]. This infall is manifest as a systematic relative motion of the Local Group with respect to distant galaxies at a velocity $v \approx 600 \text{ km s}^{-1}$, coinciding with the Local Group motion relative to the CMB frame. In order not to conflict with these observations we at least require that at redshifts $z_0 \approx 0.01$ the dipole moment due to the Dąbrowski acceleration is no larger than the observed dipole:

$$cz_1 \lesssim v = 600 \text{ km s}^{-1}.$$

Then, since at low redshift the linear Hubble law is valid, we have

$$cz_0 = H_0 r_{\text{prop}}, \quad cz_1 = \delta H r_{\text{prop}},$$

and so

$$\frac{\delta H}{H_0} = \frac{cz_1}{cz_0} \lesssim \frac{600 \text{ km s}^{-1}}{0.01c} = 0.2. \quad (45)$$

That is, we can allow at most a 20% variation in H_0 around the sky. This is comparable to the 17% variation required by the CMB limit derived above, which means that (45) does not provide a stronger constraint on the Dąbrowski model parameters than (39). Interestingly, [73,74] estimated the dipole in H_0 to be about 20% (the Rubin-Ford effect). However, this measurement has since been discredited [75], being the result of selection effects.

If the H_0 -dipole could actually be measured, or at least bounded, then the time independence of the acceleration scalar in (12) makes it a simple matter to use this to constrain the model parameters a and b . From (30) and (12) (with $\chi = \psi$)

$$\frac{\delta H}{H_0} \equiv \frac{\dot{u}}{cH_0} = \frac{2|a|}{\sqrt{\Delta}H_0} \sin \psi.$$

If $\delta H/H_0 \leq \gamma$, say, and we assume this holds for all observer positions ψ , in accordance with the Copernican principle, then

$$|a| \leq \frac{1}{2}\gamma H_0 \sqrt{\Delta}.$$

It should be borne in mind that throughout the preceding discussion we have only considered the modulus of the dipole, not its direction. It is clear from (35) that the local dipoles may be in the same direction as the CMB dipole or in the opposite direction, depending on the sign of $\sin \chi_{\text{CMB}}$. What is more, the variation of the dipole with distance is controlled entirely by $\sin \chi$, so the dipole will change sign whenever χ is a multiple of π . The alignment of the CMB and local dipoles is usually taken to be a strong sign that both result from the peculiar motion of the Local Group (due largely to GA infall). However, it is clear that for the models we consider these dipoles will also always be aligned (or anti-aligned).

F. The Combined Exclusion Diagrams

When we combine all of the constraints derived in this section (Figs. 9 and 10) we can see that for $H_0 = 50 \text{ km s}^{-1} \text{ Mpc}^{-1}$ the strongest constraint comes from the CMB, with age placing somewhat weaker limits on the allowable degree of inhomogeneity (which is measured largely by the size of a – see Sec. IV). The ‘size’ restriction of Sec. III C eliminates quite a large region of parameter space for small b , but this is not really a constraint on the inhomogeneity, which is our principal concern.

Perhaps rather surprisingly, given the results of [32,35], the strongest constraint for larger H_0 really comes from the age. As can be seen from the exclusion plot for $H_0 = 70 \text{ km s}^{-1} \text{ Mpc}^{-1}$, Fig. 10, the CMB constraint pokes out in places to eliminate certain regions, and the size constraint cuts off low values of b , but age does most of the dirty work. It can also be seen that for $H_0 = 80 \text{ km s}^{-1} \text{ Mpc}^{-1}$ age imposes a very strong constraint on the models (dashed line in Fig. 10): models with high H_0 must be very nearly homogeneous, or they are too young. However, if we relaxed the age constraint to 10 Gyr the CMB anisotropy would be the dominant limitation for most values of H_0 in the currently fashionable range ($50 \lesssim H_0 \lesssim 80 \text{ km s}^{-1} \text{ Mpc}^{-1}$).

We should not forget, at this point, to reintroduce the restriction (20) from the dominant energy condition, which rules out high b . This is not shown on the diagrams, in order to avoid clutter. Most of the models eliminated by this constraint have already been ruled out by the age or CMB constraints, and models that are rejected solely by the dominant energy condition are not hugely inhomogeneous (see the next section).

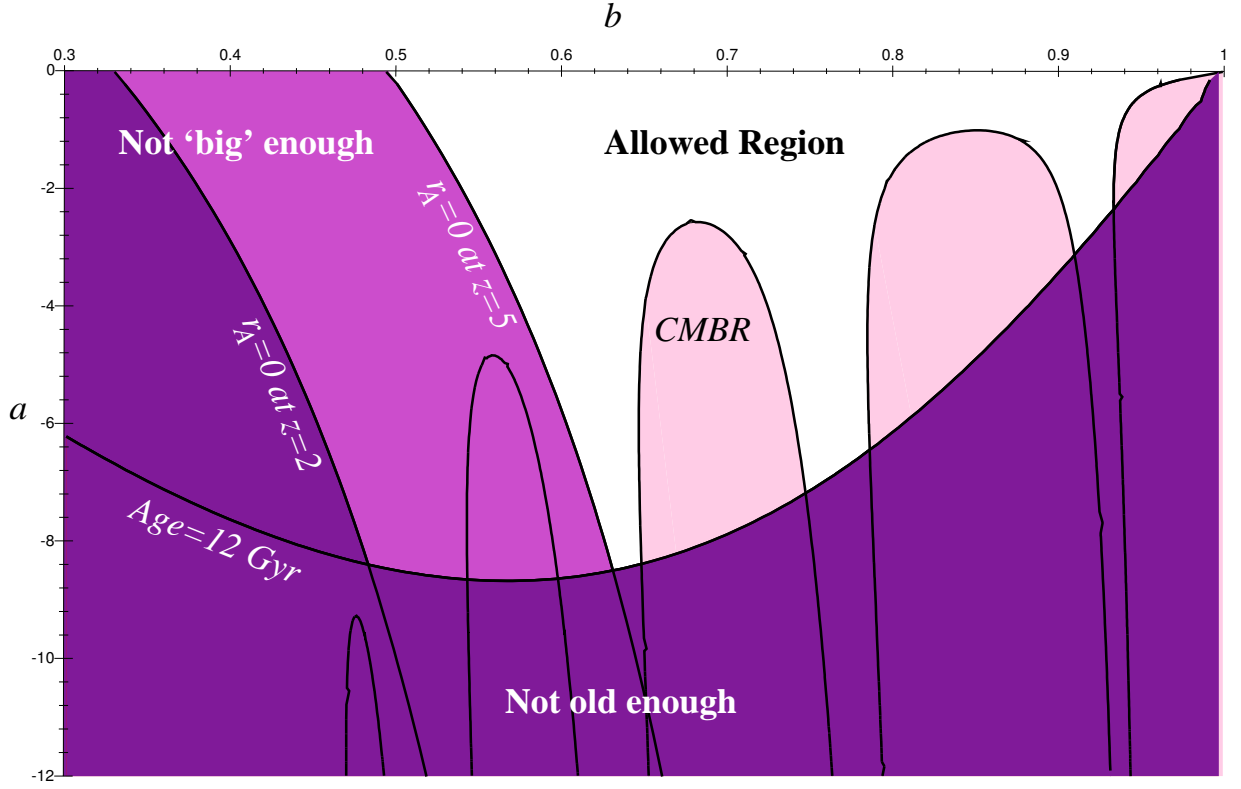


FIG. 9. The complete exclusion diagram for all the observational constraints studied (age, size and the CMB anisotropy), for $H_0 = 50 \text{ km s}^{-1} \text{ Mpc}^{-1}$. We have taken the 12 Gyr age constraint, to be conservative. The dominant energy condition should be added to these constraints: it eliminates models with $b > 0.82$ (equation (20)).

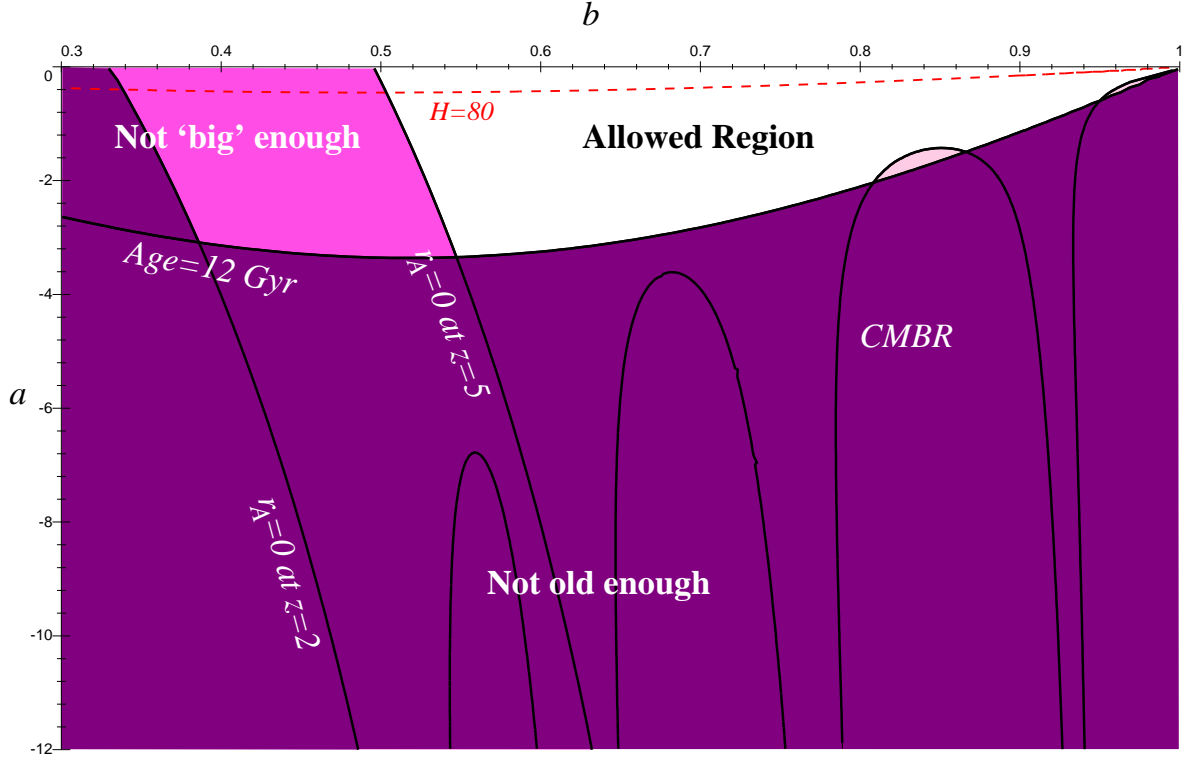


FIG. 10. As in Fig. 9, but for $H_0 = 70 \text{ km s}^{-1} \text{ Mpc}^{-1}$. The age constraint for $H_0 = 80 \text{ km s}^{-1} \text{ Mpc}^{-1}$ is also shown as a dashed line (close to the b -axis): high H_0 means a low age.

IV. THE SIZE OF THE INHOMOGENEITY

While we have considered many different aspects of the Dąbrowski models, what we have not done is to assess the extent to which the models that are not excluded are inhomogeneous. It is obvious from the exclusion plots, Figs. 9 and 10, that the homogeneous Dąbrowski models (those with $a = 0$) are the ‘most acceptable’, in that all the constraints favour small a . This should not be surprising, as far as anisotropy constraints are concerned, at least. What is not clear is whether the allowed region only contains models that are very nearly homogeneous. We will show that it does not.

The most natural way to assess the degree of inhomogeneity of the models is to examine the variation of the pressure over surfaces of constant cosmic time. It can be seen from (17) that the extremes of pressure occur at $\chi = 0$ and $\chi = \pi$, so we define the inhomogeneity factor Π to be the relative pressure difference between the two poles:

$$\Pi = \left| \frac{p(\pi) - p(0)}{p(0)} \right| = \frac{8|a|R}{c\Delta}. \quad (46)$$

If $\Pi \gtrsim 1$ then it is reasonable to say that the models are truly inhomogeneous, whereas if $\Pi \ll 1$ they are obviously nearly FLRW. Note, though, that Π depends on the cosmic time surface under consideration: for small t , $\Pi \approx 0$, and Π reaches its maximum at $t = -b/2a$ (see (4)), at which time

$$\Pi = \frac{2b^2}{1 - b^2} = 2 \frac{1 - \Delta}{\Delta}$$

so that the models are significantly inhomogeneous ($\Pi \gtrsim 1$) when

$$b \gtrsim \frac{1}{\sqrt{3}} \approx 0.58. \quad (47)$$

Most of the allowed models in Figs. 9 and 10 satisfy (47) – models with smaller b have already been eliminated by the size constraint in Sec. III C.

This is not really a fair reflection of the inhomogeneity of the models at the times of observation that are relevant here, though, because the H_0 constraint (27) generally ensures that the epoch of observation is quite early on in the evolution of the universe when the scale factor is somewhat smaller than its maximum size. To evaluate the impact of this, consider two specific examples. From the allowed region of Fig. 9 choose the model at $a = -7$, $b = 0.75$. For $H_0 = 50 \text{ km s}^{-1} \text{ Mpc}^{-1}$ the solution of (27) is $T \approx 0.016 \text{ Mpc s km}^{-1}$, which gives $\Pi = 1.33$. For the model at $a = -8$, $b = 0.64$ we get $T = (3 - \sqrt{5})/50 \approx 0.015 \text{ Mpc s km}^{-1}$ and $\Pi = 0.86$, which is close enough to 1. These models are certainly not ‘close to FLRW’.

Nevertheless, it could be said that the models are not massively inhomogeneous, and that their degree of inhomogeneity only reflects the looseness of the constraints applied. This is not so: even for models that are only inhomogeneous at the 10% level ($\Pi = 0.1$), the CMB anisotropy, $\delta_1 < 10^{-4}$, is at least *three orders of magnitude smaller* than the inhomogeneity that generates it. This certainly conflicts with the spirit of the almost EGS theorem of [5] which says that small CMB anisotropies indicate correspondingly small perturbations from homogeneity. Of course, we know from [1] that the Dąbrowski models all *admit* a perfectly isotropic radiation field; the fact that the CMB is not precisely isotropic here is a result of our more sophisticated definition of the CMB surface.

V. A NOTE ON MODELS WITH $a > 0$

In this paper we have only considered the effects of the models with $a \leq 0$ in order to satisfy the strong energy condition. The type Ia supernovae data strongly implies that the expansion rate of the universe is increasing ($q_0 < 0$) which, in the $\Lambda = 0$ FLRW case at least, can only occur when the strong energy condition fails, as may be seen from the Raychaudhuri equation. This is not necessarily true when acceleration is present however [6,8], but we may derive the following condition for the Dąbrowski models to have $q_0 < 0$ at some time t :

$$\frac{a}{\Delta} \left[\Delta + ((2at + b)^2 + b^2) \sin^2 \frac{\psi}{2} \right] > 0.$$

The term in brackets is always positive (unless $\Delta < 0$, which we do not consider), so that $q_0 < 0$ if and only if $a > 0$. In fact, this is intuitively clear from the quadratic form of $R(t)$: the accelerated expansion implied by the SNIa data can only be produced when $R(t)$ is an ‘upright’ quadratic ($a > 0$).

By analogy with FLRW models, we may define a Hubble normalised density parameter

$$\Omega(t) = \frac{24\pi G\mu}{\Theta^2} = (2at + b)^{-2},$$

which we may use to get a rough idea of how these models behave in comparison with FLRW models. The present day value of the density parameter is roughly in the range $0.3 \lesssim \Omega_0 \lesssim 1$. For the Dąbrowski models, we note that in the limit $t \rightarrow 0$ we have $\Omega \rightarrow 1/b^2$, and $\Omega_{,t} \rightarrow -4a/b^3$. We know that b is positive in order that the scale factor is positive, which implies that $\Omega_{,t}$ is negative at the big bang only if $a > 0$. However, we also see that as $t \rightarrow 0$, then Ω approaches some value larger than 1 if $b < 1$ (which is required if we reject models with finite-density singularities). This means that if we desire $\Omega \lesssim 1$ today, then we need $a > 0$, in contradiction with the strong energy condition.

When $a > 0$, the quantitative restrictions from the age, size and CMB anisotropies are not strongly affected; indeed, in some ways the outlook is better. As is shown in [6,35], the proper age of an observer in an $a > 0$ model is considerably larger than their compatriot in a negative a model, for the same value of H_0 ; the ‘size’ of the universe (i.e., the redshift of the antipode) is also larger for $a > 0$. The CMB anisotropy goes as $|z_1| \sim |a|$, which suggests that it is unaffected by the sign of a .¹

However, $a > 0$ implies that the universe will open up at $t = (1 - b)/2a$, after which time the universe is infinite in spatial extent (note that models with $a > 0$ expand forever). Spatial infinity occurs at some finite $\psi = \psi_{max}$, where $W = 0$ in (10). Applying the Copernican principle in this case becomes difficult – at least in terms of producing exclusion diagrams. We will not give further attention to this here.

VI. CONCLUSIONS

We have studied the physical, geometrical and observational characteristics of a two-parameter family of inhomogeneous, perfect fluid cosmological models which form a subclass of the models admitting isotropic radiation fields found in [1] and coincide with the model I spacetimes of [34]. We have shown that these models do not suffer from particle horizons.

The inhomogeneity of these Dąbrowski models makes the investigation of their observational characteristics from any position essential, and the simple conformal geometry of the models was used in Secs. II-III to derive exact, analytic expressions for the observational relations and other properties for any observer position; the constraints imposed by observations were examined. For any model parameters and any observer position we fixed the *coordinate* time T of the present epoch (i.e., the time of observation) by demanding that the value of H_0 at that T , given by (27), was consistent with current estimates of the Hubble constant (28). We then proceeded to test the consistency of the models with a variety of observational constraints, the most important being age and the anisotropy of the CMB.

Obviously, in any viable cosmological model the age of the universe must be greater than the estimated age of any of its components by the time the expansion has slowed to the rate we measure today. This is position dependent in our models: on a given surface of constant coordinate time (determined by the expansion rate H_0) the observers at the centre will generally be older than their counterparts elsewhere (see Fig. 1). As with all the tests we apply, we show the constraints for the ‘worst case’ observer position, i.e., that which excludes the largest region of parameter space.

Despite the fact that the Dąbrowski models *admit* an isotropic radiation field [1], this will not in general correspond exactly to the actual cosmic background radiation field after decoupling. This is because, although the models are homogeneous at early times (conformal factor $W = 1$), the inhomogeneity ($W = W(\chi, t)$) that develops up to the time of decoupling will leave its imprint in the CMB. In other words, the EGS-type theorems prove the existence of an isotropic radiation field, but whether this is realised in a particular spacetime depends on the physics of decoupling. We used a pragmatic definition of the time of decoupling to estimate the effect of the inhomogeneity on the CMB. The anisotropies in the observed CMB temperature go as $1/(1+z) \sim 1/W$: that is, all anisotropies arise entirely from the inhomogeneities in the conformal factor W *at the time of decoupling*; the conformal flatness of the spacetimes allows the light from the CMB surface to travel unmolested by any subsequent deviation from homogeneity, and we see the CMB as a reflection of the universe’s inhomogeneity at that time. Since t is small at decoupling $W \sim 1$ and the inhomogeneities are very small. The dipole moment of the CMB is the largest moment, which we restrict to be

¹This is not quite true, however, because it depends on the time at which the surface of last scattering occurs (i.e., on χ_{CMB}). This will be later for models with $a > 0$ (since the epoch of observation is later, for the same H_0), and the universe will be more inhomogeneous. However, this effect turns out to be rather weak and does not affect the conclusions we reach.

smaller than 10% of the observed dipole, on the basis that present data cannot show that Local Group motion and the CMB dipole agree to within 10% [72].

In addition to these observational constraints, we required that the first zero of the distance-redshift relations does not occur too close to the observer, on the grounds that this would probably have been observed. This ruled out models with small b , which turned out to be fairly homogeneous anyway.

The matter content of these models is unusual, in that there is no equation of state. However, we know from [1] (see also [6,38]) that they do admit a thermodynamic interpretation (i.e., definitions of number density, temperature and entropy throughout spacetime consistent with the first law of thermodynamics). We have shown that many, though not all, of the Dąbrowski models satisfy the weak, strong and dominant energy conditions (Sec. IID), so that their matter content cannot be ruled out as obviously unphysical. We restricted our detailed analysis to the models that satisfy all three energy conditions (the strong energy condition then requires $a < 0$). Of course, we are not forced to accept the energy conditions. Although they seem physically very reasonable conditions to impose on any form of matter there are many examples in cosmology of matter that does not satisfy them (quantum fields, for example, can exhibit negative energy density). Given that quintessence models [44–47], which produce an accelerating expansion in FLRW models (consistent with the supernova data), must break the strong energy condition, rejection of the $a > 0$ models may be premature, and we have considered their properties briefly in Sec. V (see also [6]).

Our studies have shown that there is a significant subset of the Dąbrowski models that are markedly inhomogeneous but cannot be excluded on the basis of the tests considered here. It is possible, *for every observer* in each of the models in the allowed regions of Figs. 9 and 10, to choose the epoch of observation so that the observed value of H_0 is reproduced, the age is greater than the measured age of the universe and there are no obviously unacceptable features at low redshift ($z \lesssim 5$) in the observational relations. Most importantly, though, the dipole in the cosmic background radiation would be considerably smaller than the observed CMB dipole: despite the inhomogeneity of the models the anisotropy they produce is very small. The fact that this is true for every observer means that it is not possible to reject these models by appealing to the Copernican principle. As a result, the standard assumption that the observed high degree of isotropy about us combined with the Copernican principle necessarily forces the universe to be homogeneous (i.e., the cosmological principle) is seriously undermined.

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APPENDIX A: TRANSFORMATION TO A NON-CENTRAL POSITION

We want to transform the metric (9) from the (χ, θ, ϕ) coordinate system, whose origin is at the centre, to coordinates centred instead on some observer at $\chi = \psi$, while preserving the form of the FLRW part of the metric. It is therefore necessary to identify the transformations of the (homogeneous) FLRW spatial sections that leave the FLRW metric invariant, i.e., the isometries of the spatial sections. This is simple. Since the spatial sections of an FLRW model with positive curvature constant ($\Delta > 0$) are 3-spheres, the isometries we require are 4-dimensional rotations (i.e., elements of $SO(4)$, the isometry group of the 3-sphere).

A sphere of radius R in 4-dimensional space with cartesian coordinates (x, y, z, u) is defined by

$$x^2 + y^2 + z^2 + u^2 = R^2.$$

We have three coordinates on this sphere: χ and the two spherical polar angles θ and ϕ . These are related to the cartesian coordinates by

$$x = R \sin \chi \sin \theta \cos \phi \tag{A1}$$

$$y = R \sin \chi \sin \theta \sin \phi \tag{A2}$$

$$z = R \sin \chi \cos \theta \tag{A3}$$

$$u = R \cos \chi \tag{A4}$$

The origin, $\chi = 0$, is then at $x = y = z = 0$, $u = R$. We are only interested in 4-rotations that move the origin, and, as the initial metric is spherically symmetric (really spherically symmetric, not just conformally: even the conformal

factor is spherically symmetric about the centre), we need only consider moving the observer in one direction, which we choose to be the z direction (i.e., to a position with non-zero z , but $x = y = 0$). Clearly, then, we are looking for a rotation in the $u - z$ plane. Since we have the conformal factor as a function of χ we want to find χ as a function of the new coordinates. Starting with coordinates χ', θ' and ϕ' , centred on some position $\chi = \psi$, along with their primed cartesian counterparts x', y', z' and u' (which are related in the same way as the unprimed coordinates in (A1)–(A4)), a rotation back to the original coordinates is given, in cartesian coordinates, by $x = x', y = y'$ and

$$\begin{aligned} z &= \cos \psi z' + \sin \psi u', \\ u &= -\sin \psi z' + \cos \psi u'. \end{aligned} \quad (\text{A5})$$

(Note that at the origin of the primed coordinates, where $z' = 0$ and $u' = R$, we have $u = R \cos \psi$, showing that $\chi = \psi$ there, as required.) Equation (A5), along with the primed versions of (A3) and (A4), then immediately gives

$$\cos \chi = \cos \psi \cos \chi' - \sin \psi \sin \chi' \cos \theta', \quad (\text{A6})$$

and this is all we will need, since the only spatial coordinate that enters into the original metric (9) is χ , and that enters only as $\cos \chi$ ($2 \sin^2 \frac{\chi}{2} = 1 - \cos \chi$).

APPENDIX B: REDSHIFT IN CONFORMALLY RELATED SPACETIMES

We present two derivations of the redshift formula for spacetimes sharing some of the simple properties of the Dąbrowski models. The first can be used in spacetimes that are conformal to simpler metrics for which the geodesics and redshifts can be found. The second is valid for spacetimes that are conformal to a spherically symmetric spacetime.

If we have two conformally related metrics, $g_{ab} = \Omega^2 \bar{g}_{ab}$ ($\Omega > 0$), their associated metric connections are related by (see appendix D of [40])

$$\nabla_b V^a = \bar{\nabla}_b V^a + (\bar{\nabla}_b \ln \Omega) V^a + (V^c \bar{\nabla}_c \ln \Omega) \delta_b^a - (\bar{\nabla}_d \ln \Omega) g^{ad} g_{bc} V^c, \quad (\text{B1})$$

for any vector field V^a . Null geodesics with respect to g (or, more correctly, with respect to ∇) satisfy $g_{ab} k^a k^b = 0$ and

$$k^b \nabla_b k^a = 0, \quad (\text{B2})$$

where k^a is the tangent vector to the geodesic. Applying (B1) gives

$$0 = k^b \nabla_b k^a = k^b \bar{\nabla}_b k^a + 2k^b \bar{\nabla}_b \ln \Omega k^a - g^{ad} \bar{\nabla}_d \ln \Omega (g_{bc} k^b k^c) = \frac{1}{\Omega^4} (\Omega^2 k^b) \bar{\nabla}_b (\Omega^2 k^a),$$

and we see immediately that

$$\bar{k}^a = \Omega^2 k^a \quad (\text{B3})$$

is the tangent vector to a null geodesic with respect to $\bar{\nabla}$. So, every null geodesic of ∇ corresponds to a null geodesic of $\bar{\nabla}$ (and vice versa, since we can repeat the above steps interchanging g and \bar{g} and putting $\Omega \mapsto 1/\Omega$). If the geodesics with respect to $\bar{\nabla}$ are known explicitly then we can find them easily for ∇ .

To find the redshift, though, we also need the velocities of emitter and observer. A four-velocity satisfies $g_{ab} u^a u^b = -c^2$, and if we define

$$\bar{u}^a = \Omega u^a \quad (\text{B4})$$

then \bar{u} is a four-velocity with respect to \bar{g} : $\bar{g}_{ab} \bar{u}^a \bar{u}^b = -c^2$. Redshift is calculated from the ratio of the emitted frequency $\nu_E = u_a k^a|_E$ to the observed frequency $\nu_O = u_a k^a|_O$. Using (B3) and (B4) we have

$$u_a k^a = g_{ab} u^a k^b = \Omega^2 \bar{g}_{ab} \frac{\bar{u}^a \bar{k}^b}{\Omega \Omega^2} = \frac{1}{\Omega} \bar{u}_a \bar{k}^a,$$

which means that

$$1 + z = \frac{u_a k^a|_E}{u_b k^b|_O} = \frac{\Omega_O}{\Omega_E} \frac{\bar{u}_a \bar{k}^a|_E}{\bar{u}_b \bar{k}^b|_O} = \frac{\Omega_O}{\Omega_E} (1 + \bar{z}). \quad (\text{B5})$$

where \bar{z} is the redshift associated with \bar{g}_{ab} for the fundamental velocity \bar{u} . If the paths of null rays in the spacetime \bar{g}_{ab} and the redshift formula for the velocity \bar{u} are known then (B6) gives the redshift in the true spacetime. For the Dąbrowski models \bar{g}_{ab} is an FLRW metric, the conformal factor is $\Omega = 1/W$ (see (9)) and \bar{u} is the usual FLRW comoving velocity field. The well-known expression for redshift in FLRW spacetimes, $1 + \bar{z} = R_O/R_E$, then gives

$$1 + z = \frac{R_O}{W_O} \frac{W_E}{R_E}. \quad (\text{B6})$$

When the true spacetime is conformal to a spherically symmetric spacetime the radial null geodesics connecting any point with an observer at the centre are obviously purely radial (since their paths are not affected by the conformal factor). They are therefore given (in terms of coordinates r and t with respect to which the spherical symmetry is manifest) by some function $t_O(r_E, t_E)$ relating the time, t_O , that the light ray is received by the observer, to the time of emission, t_E , for an object at radius r_E . This is just the lookback-time relation. Redshift, as the ratio of proper time intervals $d\tau_O$ at the observer to proper time intervals $d\tau_E$ at the emitter, is then given by

$$1 + z \equiv \frac{d\tau_O}{d\tau_E} = \frac{d\tau_O}{dt_O} \frac{dt_O}{d\tau_E} = \frac{d\tau_O}{dt_O} \left(\frac{\partial t_O}{\partial r_E} \frac{dr_E}{d\tau_E} + \frac{\partial t_O}{\partial t_E} \frac{dt_E}{d\tau_E} \right) = \frac{1}{u_O^t} \left(\frac{\partial t_O}{\partial r_E} u_E^r + \frac{\partial t_O}{\partial t_E} u_E^t \right). \quad (\text{B7})$$

(When the coordinates r and t are comoving – $u^r = 0$ – the r -derivative term disappears.) This will provide an analytic expression for the redshift whenever the lookback-time equation can be integrated. For the Dąbrowski models:

$$u^x = 0, \quad u^t = \frac{c}{|g_{00}|^{1/2}} = W \quad (\text{B8})$$

and the lookback time can be derived directly from the metric: on the past null cone of the observer $ds = 0 = d\theta = d\phi$, leading to an expression for $d\chi/dt$, which, when integrated, gives

$$\chi = c\sqrt{\Delta} \int_t^T (\chi, t) \frac{dt'}{R(t')}.$$

Differentiating this with respect to t at fixed χ then gives

$$\frac{\partial t_O}{\partial t_E} \equiv \frac{\partial T}{\partial t} = \frac{R_O}{R_E},$$

which, together with (B7) and (B8), results in the expression (B6) for the redshift.

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